Numerical Programming 2 (CSE) 2015

Worksheet 2

Exercise 1

Classify the following linear second order partial differential equations defined on the x-y plane

- $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$
- $(x^2 - 1)u_{xx} - xu_x + 2xyu_{xy} + (y^2 - 1)u_{yy} + yu_y = 0$
- $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

Exercise 2

- Given a set of orthonormal functions $\{\phi_1(x), ..., \phi_N(x)\}$, $\phi_i(x) = 0$, show that the Gram matrix

  $$G_{ij} = \int_\Omega \nabla\phi_i(x) \cdot \nabla\phi_j(x) dx$$

  introduced in the context of the Galerkin method is symmetric positive definite (therefore invertible) independently of the choice of the basis $\{\phi_i\}_{i=1}^N$.

- Consider the one dimensional boundary value problem:

  $$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + b(x)u(x) = f(x) \qquad x \in (a, b)$$

  with assigned boundary condition $u(a) = u_a$ and $u(b) = u_b$. Find the weak formulation of this problem using integration by parts and test functions v vanishing on the boundary $v(a) = v(b) = 0$.

Exercise 3

Let $\Phi^{t,H}$ be the flow map associated to the ODE $\dot{z} = \Omega\nabla_z H$, which is induced by the Hamiltonian H.

Noether's theorem states that for two Hamiltonians H and G,

$$G\left(\Phi^{t,H}z\right) = G(z)\ \forall t, z \ \Leftrightarrow\ H\left(\Phi^{t,G}z\right) = H(z)\ \forall t, z,$$

i.e. a quantity G is conserved by the flow of H, if and only if H is invariant with respect to $\Phi^{t,G}$.

- Prove Noether's Theorem.

  Hint: Compare $\frac{\mathrm{d}}{\mathrm{d}t} G\left(\Phi^{t,H} z\right)$ and $\frac{\mathrm{d}}{\mathrm{d}t} H\left(\Phi^{t,G} z\right)$.

- Use Noether's Theorem to show that the three dimensional two-body system given by the Hamiltonian

$$H(z) = \frac{1}{2m_1} \sum_{i=1}^{3} \left(p_i^{(1)}\right)^2 + \frac{1}{2m_2} \sum_{i=1}^{3} \left(p_i^{(2)}\right)^2 + \frac{m_1 m_2}{\left\| q^{(1)} - q^{(2)} \right\|},$$

  with $z = (q^{(1)}, q^{(2)}, p^{(1)}, p^{(2)})$, conserves

$$G(z) = p_1^{(1)} + p_1^{(2)},$$

  the linear momentum in x_1-direction.

  Hint: You do not need to compute any derivatives of H.