Numerical Programming 1 (CSE) 2014

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Solutions for Worksheet 7

Exercise 1

- Let v_0 be an eigenvector for λ_{max}: $Av_0 = \lambda_{max} v_0$. If $\|\cdot\|$ is a matrix norm which is induced by a vector norm, i. e. $\|A\| = \max_{v\neq 0} \frac{\|Av\|}{\|v\|}$, then we show the inequality as follows:

$$\|A\| = \max_{v\neq 0} \frac{\|Av\|}{\|v\|} \geq \frac{\|Av_0\|}{\|v_0\|} = \frac{\|\lambda_{max} v_0\|}{\|v_0\|} = |\lambda_{max}|.$$

  For general matrix norms (which may not be induced by a vector norm), it is still possible to show the inequality: Consider the matrix $V_0 = (v_0|v_0|\dots|v_0)$ which repeats the column vector v_0. Then one has $AV_0 = \lambda_{max} V_0$ and thus

$$\|A\|\|V_0\| \geq \|AV_0\| = \|\lambda_{max} V_0\| = |\lambda_{max}|\|V_0\|,$$

  i. e. $\|A\| \geq |\lambda_{max}|$.

- By the definition of the condition number of a square matrix,

$$k(AB) = \|AB\|\|\underbrace{(AB)^{-1}}_{B^{-1}A^{-1}}\| \leq \|A\|\|B\|\|B^{-1}\|\|A^{-1}\| = k(A)k(B).$$

Exercise 2

We begin with the values of some integrals we will frequently need:

$$\frac{1}{2}\int_{-1}^{1} \begin{matrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{matrix} dx = \begin{matrix} 1 \\ 0 \\ 1/3 \\ 0 \\ 1/5 \\ 0 \\ 1/7 \end{matrix}$$

Now, let us start with the Gram-Schmidt process on $\{1, x, x^2\}$:

- Normalize $\tilde{v}_1 = 1$:

$$v_1 := \frac{\tilde{v}_1}{\|\tilde{v}_1\|} = \frac{\tilde{v}_1}{\sqrt{\langle \tilde{v}_1, \tilde{v}_1 \rangle}} = \frac{\tilde{v}_1}{1} = 1$$

- Subtract the orthogonal projection of x onto $span(v_1)$ from x to make it orthogonal on v_1:

$$\tilde{v}_2 = x - \underbrace{\langle v_1, x \rangle}_{=0} v_1 = x$$

- Normalize $\tilde{v}_2 = x$:

$$v_2 := \frac{\tilde{v}_2}{\|\tilde{v}_2\|} = \frac{x}{\underbrace{\|x\|}_{\sqrt{1/3}}} = \sqrt{3}x.$$

- Subtract the orthogonal projection of x^2 onto the plane $span(v_1, v_2)$ from x^2 to make it orthogonal on $span(v_1, v_2)$:

$$\tilde{v}_3 = x^2 - \underbrace{\langle v_1, x^2\rangle}_{=1/3} v_1 - \underbrace{\langle v_2, x^2\rangle}_{=0} v_2 = x^2 - \frac{1}{3}$$

- Normalize $\tilde{v}_3 = x^2 - \frac{1}{3}$:

$$v_3 := \frac{\tilde{v}_3}{\|\tilde{v}_3\|} = \frac{x^2 - \frac{1}{3}}{\|x^2 - \frac{1}{3}\|} = \frac{x^2 - \frac{1}{3}}{\underbrace{\sqrt{\langle x^2 - 1/3, x^2 - 1/3\rangle}}_{\sqrt{4/45}}} = (x^2 - \frac{1}{3})\frac{3}{2}\sqrt{5}.$$

The vector representation of $\{v_1, v_2, v_3\}$ in the basis $\{1, x, x^2\}$ is given by $\begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = A \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$ with $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ -\frac{1}{2}\sqrt{5} & 0 & \frac{3}{2}\sqrt{5} \end{pmatrix}$. The other way round, we have $\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} = A^{-1} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}$ with $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 1/3 & 0 & \frac{2}{3\sqrt{5}} \end{pmatrix}$. The rows of A^{-1} represent $\{1, x, x^2\}$ in the basis $\{v_1, v_2, v_3\}$.

Continue with the QR decomposition of A^{-1} e. g. with Housholder reflection: So we have to project the first row $\begin{pmatrix} 1 \\ 0 \\ 1/3 \end{pmatrix}$ to a multiple of the first unit vector $-\alpha e_1 = \begin{pmatrix} -\sqrt{\frac{10}{9}} \\ 0 \\ 0 \end{pmatrix}$. We have $v = b_1 + x = \begin{pmatrix} 1 - \sqrt{\frac{10}{9}} \\ 0 \\ 1/3 \end{pmatrix}$ and so

$$H_v = I - 2\frac{vv^T}{v^Tv} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \frac{2}{20/9 - 2\sqrt{\frac{10}{9}}} \begin{pmatrix} \left(1 - \sqrt{\frac{10}{9}}\right)^2 & & \frac{1}{3}\left(1 - \sqrt{\frac{10}{9}}\right) \\ & 0 & \\ \frac{1}{3}\left(1 - \sqrt{\frac{10}{9}}\right) & & 1/9 \end{pmatrix}$$

One can see that $R := H_v A^{-1}$ is an upper triangular matrix, so the QR decomposition is $A^{-1} = H_v^{-1} R = H_v R$.

For the minimization problem, geometrically we have to project x^3 orthogonally (w.r.t. the inner product considered in this exercise) onto the (three-dimensional) space of polynomials of degree up to 2.

$$\text{Proj}(x^3) = \langle v_1, x^3\rangle v_1 + \langle v_2, x^3\rangle v_2 + \langle v_3, x^3\rangle v_3 = \frac{3}{5}x$$

Exercise 3

$$H_v v = (I - 2\frac{vv^T}{v^T v})v = v - 2\frac{vv^T v}{v^T v} = v - 2v = -v$$

$$H_v u = (I - 2\frac{vv^T}{v^T v})u = u - 2\frac{v \overbrace{v^T u}^{=0}}{v^T v} = u \text{ for } u \text{ perpendicular to } v.$$

Geometric interpretation: The Householder reflector of v reflects at the hypersurface (dimension: n-1) $v^{\perp}$ orthogonal on v. So e. g. v is mapped to $-v$ and the set $v^{\perp}$ is fixed.

Eigenvalues: If one chooses a basis $\{v_1, v_2, \dots, v_{n-1}\}$ of $v^{\perp}$, then a basis of $\mathbb{R}^n$ is given by $\{v, v_1, v_2, \dots, v_{n-1}\}$. These are eigenvectors for the eigenvalues $-1, 1, 1, \dots, 1$. The choice of eigenvectors is not unique in this case, as we could have chosen any basis on $v^{\perp}$, but the eigenvalues of a map/matrix are unique. the determinant is given by the product of all the eigenvalues, so $-1 \cdot 1 \cdot 1 \cdot \dots \cdot 1 = -1$. This means that volumes are retained but orientations are reversed.