Numerical Programming 1 (CSE) 2014

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Worksheet 6

Exercise 1

Consider the function $f(x) = x^5$. Evaluate $\int_0^1 f(x)dx$ using Monte Carlo estimates with N pseudorandom numbers uniformely distributed on $[0,1]$ (use the Matlab function `rand`) and plot the error $\texttt{Er} = |\int f - \frac{1}{N}\sum_i f(x_i)|$ of your estimates for different values of N. In this exercise we will try to improve the error using the following equality

$$E_u[f] = \int_0^1 f(x)dx = \int_0^1 \frac{f(x)}{w(x)} w(x)dx = E_w[f/w]. \tag{1}$$

The equality is saying that you can see the integral of f as the expected value of f with respect to the probability density function

$$u(x) = \begin{cases} 1 & \text{for } \ x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

or, equivalently, as the expected value of f/w with respect to the probability density function w[1]. The equality (1) is at the base of a technique called *importance sampling*. The goal of this exercise is to convince you that a wise choice of the function w can lead to estimates with a smaller error than those ones you have seen in the first part of the exercise.

Using the law of large numbers directly on the quantity $E_w[f/w]$ leads to

$$E_w[f/w] \approx \frac{1}{N}\sum_{i=1}^{N} \frac{f(y_i)}{w(y_i)} \tag{2}$$

where y_i are points independently distributed according to w. Consider $w(x) = (k+1)x^k$ with $k \leq 5$. You can generate pseudorandom numbers y_i distributed according to w by generating first uniformely distributed points x_i on $[0,1]$ and then mapping them according to the transformation $y_i = (x_i)^{\frac{1}{k+1}}$. Fix $k = 4$ and estimate $E_w[f/w]$ as in (2) for different N. Plot the error $\widetilde{\texttt{Er}} = |\int f - \frac{1}{N}\sum_i \frac{f(y_i)}{w(y_i)}|$ for different values of N and compare it to `Er`. What do you notice? Repeat the experiment with increasing values of $k \leq 5$ (choose for instance $k = 4, 4.5, 4.7, 5$). How does the error behave as k increases? Explain.

[1] We will assume that $w(x) \geq 0$ and $\int_0^1 w(x)dx = 1$.

Exercise 2

Given $x \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$ verify the following inequalities

- $\| x \|_\infty \leq \| x \|_2$
- $\| x \|_2 \leq \sqrt{m} \| x \|_\infty$
- $\| A \|_\infty \leq \sqrt{n} \| A \|_2$
- $\| A \|_2 \leq \sqrt{m} \| A \|_\infty$

Exercise 3

Given the linear system of equations

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 1 & 0 & \alpha \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \\ 1 \\ 0 \end{pmatrix} \tag{3}$$

use the LU factorization to characterize the solution of the system as α and β change. In particular, for which values of α and β does the system have a solution? When is the solution unique? Is there a choice of α and β for which the system admits multiple solutions?