Numerical Programming 1 (CSE) 2014

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Worksheet 2

Exercise 1

Consider the function $f(x) = 1/(1+x^2)$ on the interval $[-5, 5]$. Using the Matlab function `linspace(-5,5,N)` generate N points uniformly distributed on this interval and plot different polynomial interpolations using $N = 5, 15, 25$ and 35 each time [1]. How do the interpolating polynomials compare with $f(x)$?

When using a polynomial $p(x)$ of degree n to interpolate a function $f(x) \in C^{n+1}$ at points $x_0 > ... > x_n$ the error $E(x) = p(x) - f(x)$ (for $x \in [x_0, x_n]$) can be expressed as

$$E(x) = l(x)\frac{f(\xi)^{(n+1)}}{(n+1)!} \qquad \text{where} \qquad l(x) = (x-x_0)(x-x_1)...(x-x_n). \quad (1)$$

Plot only the function $l(x)$ on the same grid for the values of N indicated above. What can you conclude?

Repeat the interpolation, but this time using N Chebyshev points as a grid (write your own function or use the Matlab function `chbpnt`[2]). Has the function $l(x)$ the same behavior as in the case of a uniform spaced grid?

Exercise 2

You have seen in class how discrete Fourier transform is defined. In particular, given $N > 0$, $\theta = \frac{2\pi}{N}$ and $w = e^{i\theta}$ you know that, if we define the set of vectors

$$v^k = [1, w^k, w^{2k}, ..., w^{(N-1)k}] \text{ for } k = 0, ..., N-1 \quad (2)$$

then

$$\langle v^k, v^l \rangle = (v^k)^* \cdot v^l = \frac{1}{N}\sum_{j=0}^{N-1} w^{-jk} w^{jl} = \delta_{kl}.$$

It is indeed possible to show that the family of vectors v^k, $k = 0, 1, ..., N-1$ forms an orthonormal basis of $\mathbb{C}^N$ (why?). This means that, given any vector $f \in \mathbb{C}^N$, you can represent it as

$$f = \sum_{k=0}^{N-1} \hat{f}_k v^k \qquad \text{where} \qquad \hat{f}_k = \langle v^k, f \rangle = \frac{1}{N}\sum_{j=0}^{N-1} f_j (\bar{v}^k)_j. \quad (3)$$

[1] A way to get the interpolating polynomial is to use the Matlab function `polyfit(x,y,d)` specifying the degree of the polynomial to be $d = N - 1$.

[2] By default `chbpnt` gives points on $[-1, 1]$. Don't forget to rescale the output so to obtain Chebyshev points defined on $[-5, 5]$.

Write a Matlab function to compute the Fourier coefficients $\hat{f}_k$ of the vectors with entries given by

- $f_j = 2 + \sin(2\pi x_j)$
- $f_j = (x_j - x_2)^2$

where $x_j = j/N$ (pick $N = 20$ for instance) with $j = 0, 1, 2, ..., N - 1$.

Exercise 3

Assume the definitions given in exercise 2. Given two vectors f and g both in $\mathbb{C}^N$, let's define their convolution $c = f * g$ as

$$c_j = \sum_{k=0}^{N-1} f_k g_{j-k}. \tag{4}$$

Show that, given $f = \sum_{k=0}^{N-1} \hat{f}_k v^k$ and $g = \sum_{k=0}^{N-1} \hat{g}_k v^k$, it is true that

$$f * g = N \sum_{k=0}^{N-1} \hat{f}_k \hat{g}_k v^k. \tag{5}$$

Therefore, argue that the Fourier coefficients of c are given by $\hat{c}_k = \hat{f}_k \hat{g}_k$.