Numerical Programming 1 (CSE) 2014

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Solutions for Worksheet 10, Exercises 1 and 3

Exercise 1

- The characteristic polynomial of A is $(2-\lambda)(2-\lambda)-(-1)(-1)=(\lambda-1)(\lambda-3)$, so the eigenvalues of A are 1 and 3. The first few steps of the power method yield $u_1=\begin{pmatrix}2\\-1\end{pmatrix}, u_2=\begin{pmatrix}5\\-4\end{pmatrix}, u_3=\begin{pmatrix}14\\-13\end{pmatrix}$, so one can guess that normalized limit of this sequence is $u_\infty=\lim_{k\to\infty}\frac{u_k}{\|u_k\|}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$. By considering Au_∞ it turns out that u_∞ is an eigenvector of the eigenvalue 3, i.e. of the greatest eigenvalue.

- In general, the ratio between the greatest two absolute values of the eigenvalues of the matrix decides over the convergence of the power iteration. If this is large, convergence is fast. If it is close to 1, it is slow. If it is exactly 1, there is no convergence. Example: $A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, u_0=\begin{pmatrix}1\\-1\end{pmatrix}$.

Exercise 3

- The QR decomposition is

$$A=\begin{pmatrix}\cos(\theta)&\sin(\theta)\\\sin(\theta)&0\end{pmatrix}=\begin{pmatrix}\cos(\theta)&-\sin(\theta)\\\sin(\theta)&\cos(\theta)\end{pmatrix}\begin{pmatrix}1&\cos(\theta)\sin(\theta)\\0&-\sin(\theta)^2\end{pmatrix}$$

and

$$RQ=\begin{pmatrix}1&\cos(\theta)\sin(\theta)\\0&-\sin(\theta)^2\end{pmatrix}\begin{pmatrix}\cos(\theta)&-\sin(\theta)\\\sin(\theta)&\cos(\theta)\end{pmatrix}$$
$$=\begin{pmatrix}\cos(\theta)(1+\sin(\theta)^2)&\overbrace{-\sin(\theta)(1-\cos(\theta)^2)}^{-\sin(\theta)^3}\\-\sin(\theta)^3&-\sin(\theta)^2\cos(\theta)\end{pmatrix}.$$

So after the first iteration, the lower left entry (and also the upper right entry as A is symmetric here) changes from $\sin(\theta)$ to $-\sin(\theta)^3$. This gives a hint that this entry will convergence to 0 with cubic speed.

- The eigenvalues are 1 and -1. The QR method does not work here because A is already orthogonal, so the QR decomposition is $A=QR=AI$, and $RQ=IA=A$, so nothing changes.