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Exercise 1 (Spectral Methods)

(a) Solve the PDE

$$y'' + a(x)y' + b(x)y = f(x) \qquad \text{on } (-1,1)$$

with periodic boundary conditions $y(-1) = y(1)$ for $a(x) = f(x) = \sin(2\pi x)$, $b(x) = \cos(\pi x)$ via the spectral method presented in the lecture and plot the solution for $N = 4$.

(b) Estimate the convergence rate of the spectral method numerically.

Solution:
The Fourier coefficients of a, b and f (computed by hand, by Mathematica, WolframAlpha or ...) are:

$$a_2 = f_2 = -\frac{i}{2}, \qquad a_{-2} = f_{-2} = \frac{i}{2}, \qquad b_1 = b_{-1} = \frac{1}{2}$$

and 0 otherwise. Plugged into the equation

$$-n^2\pi^2\hat{y}_n + \sum_{m=-\infty}^{\infty} (i\pi m\hat{a}_{n-m} + b_{n-m})\hat{y}_m = \hat{f}_n$$

we obtain:

$$\begin{aligned} & i\pi(n-2)a_2\hat{y}_{n-2} + b_1\hat{y}_{n-1} - n^2\pi^2\hat{y}_n + b_{-1}\hat{y}_{n+1} + i\pi(n+2)a_{-2}\hat{y}_{n+2} &&= \hat{f}_n \\ \text{i.e.} \quad & \frac{\pi}{2}(n-2)\hat{y}_{n-2} + \frac{1}{2}\hat{y}_{n-1} - n^2\pi^2\hat{y}_n + \frac{1}{2}\hat{y}_{n+1} - \frac{\pi}{2}(n+2)\hat{y}_{n+2} &&= \hat{f}_n \end{aligned}$$

Truncated to $n, m = -\frac{N}{2}, \ldots, \frac{N}{2}$, this is a linear system of the form:

$$\begin{pmatrix} \ddots & \ddots & \ddots & & & & & & & \\ \ddots & \ddots & \ddots & \ddots & & & & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \frac{\pi}{2}(n-2) & \frac{1}{2} & -n^2\pi^2 & \frac{1}{2} & -\frac{\pi}{2}(n+2) & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & & \ddots & \ddots & \ddots & \ddots \\ & & & & & & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \hat{y}_{-N/2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \hat{y}_n \\ \vdots \\ \vdots \\ \vdots \\ \hat{y}_{N/2} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ i/2 \\ 0 \\ 0 \\ 0 \\ -i/2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$