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The exercises are to be handed in on Wednesday, 30.05.2012, before the tutorial.

Exercise 1 (Hamiltonian Systems)
Consider the (1-dimensional) ODE

$$x''(t) = cos(x(t)) \quad , \quad x(0) = 1,\, x'(0) = 0,$$

(a) Find the corresponding Hamiltonian function H and rewrite the ODE in its Hamiltonian form.

(b) Apply the implicit midpoint rule and the Matlab solver `ode45` on the time interval $[0, 20]$ to the problem and check for energy conservation.

(c) Choose 500 points on the circle $\{(p, q) \in \mathbb{R}^2 : \sqrt{p^2 + q^2} = 0.4\}$ in phase space as initial values of the upper ODE. Determine the evolution of this circle in time by plotting the evolved points (in phase space) at times $\mathtt{T} = \mathtt{linspace}(0, 2 * \mathtt{pi}, 8)$ (for the solutions of the ODEs take a method of your choice).

Exercise 2 (Implicit Midpoint Rule)
Consider the implicit midpoint rule:

$$y_{n+1} = y_n + hf\left(t_n + \frac{h}{2}, \frac{1}{2}(y_n + y_{n+1})\right).$$

applied to a Hamiltonian ODE $y' = J^{-1}\nabla H(y)$.

(a) Prove:

$$X := \frac{\partial y_{n+1}}{\partial y_n} = (Id + JA)^{-1}(Id - JA).$$

where $A := \frac{h}{2} D^2 H\left(\frac{y_n + y_{n+1}}{2}\right)$.

(b) Deduce the two representations

$$X = J(A + J)(A - J)^{-1}J \qquad \text{and} \qquad X = -(A - J)^{-1}(A + J).$$

Hint: You may use the fact that the matrices $(Id - JA)^{-1}$ and $(Id + JA)$ commute.

(c) Using the two representations from (b) show:

$$X^T J X = J.$$

Explain, why a numerical method with this property is called symplectic.

Exercise 3 (Finite Differences)

Consider the averaging operator $\mathcal{Y}_0$ defined by

$$(\mathcal{Y}_0 z)_k = \frac{1}{2}\left(z_{k+\frac{1}{2}} + z_{k-\frac{1}{2}}\right).$$

Prove the following identities:

$$(\mathcal{Y}_0 \Delta_0 z)_k = \frac{1}{2}\left(z_{k+1} - z_{k-1}\right) \qquad \text{and} \qquad \mathcal{Y}_0 = \left(\mathcal{I} + \frac{1}{4}\Delta_0^2\right)^{1/2}.$$