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Exercise 1 (Newton iteration)

(a) Write a MATLAB type pseudocode for a Newton iteration of a function `f` with derivative `df` starting in `x0` $=$ `0` and terminating after `N` $=$ `100` iteration steps.

(b) What does the Newton iteration compute?

(c) How large will the error be if you apply your code to the function $f(x) = 5x + 2$.

Exercise 2 (Interpolation)

Given $a \leq x_0 < \cdots < x_n \leq b$ and $f \in C[a,b]$.

(a) Derive the formula for the first barycentric interpolation from

$$p_n(x) = \sum_{j=0}^{n} f(x_j)\ell_j(x) \qquad (*)$$

(ℓ_j denotes the jth Lagrange polynomial).

(b) Write a MATLAB type pseudocode for it and explain its advantages over the direct evaluation of $(*)$.

Exercise 3 (Quadrature)

Let $I(f) = \int_a^b f(x)dx$ and $a \leq x_0 < \cdots < x_n \leq b$.

Show that the quadrature formula $Q(f) = \sum_{j=0}^n w_j f(x_j)$ integrates all polynomials up to degree n exactly, i.e.

$$Q(p) = I(p) \text{ for all } p \in \mathbb{P}_n,$$

if $w_j = \int_a^b \ell_j(x)dx$ for all $j = 0, \dots, n$, where ℓ_j denotes the jth Lagrange polynomial.

Exercise 4 (Linear systems)

(a) Write a MATLAB type pseudocode for the Jacobi iteration of a linear system $Ax = b$, $A \in \mathbb{R}^{n \times n}$, starting in $\texttt{x0} = \texttt{zeros}(\texttt{n}, \texttt{1})$ and terminating after $\texttt{N} = \texttt{100}$ iteration steps.

(b) Which convergence rate do you expect for the matrix $A = \begin{pmatrix} 1 & 0.2 & 0.3 \\ 0.1 & 1 & 0 \\ 0 & 0.4 & 1 \end{pmatrix}$?

Exercise 5 (Least squares problems)

Given a matrix $A \in \mathbb{C}^{n \times n}$ and an approximate eigenvector $v \in \mathbb{C}^n$.

Formulate the least squares problem approximating the corresponding eigenvalue and solve it via normal equations. How is the solution called?

Exercise 6 (Eigenvalue problems)
Let $A \in \mathbb{C}^{n \times n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$, $|\lambda_1| > |\lambda_n| > \cdots > |\lambda_n|$.

(a) Write a MATLAB type pseudocode performing an inverse iteration (without shift) with a random initial vector terminating if the residual norm is less than 10^{-8}.

(b) What is computed by your code and what does the Rayleigh quotient of the resulting vector approximate?