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All exercises are to be handed in on Thursday, 17.01.2011, before the lecture.

Exercise 1 (Orthogonal Projections)
A projector is a square matrix $P \in \mathbb{R}^{n\times n}$ that satisfies $P^2 = P$.
An orthogonal projector is a projector P that satisfies $Px \perp x - Px$ (i.e. $\langle Px, x - Px\rangle = 0$) for all $x \in \mathbb{R}^n$.

(a) Prove: A projector P is orthogonal, if $P = P^T$ (in fact, the converse is true as well).

(b) Let $a \in \mathbb{R}^n$, $a \neq 0$. Show that P_a is an orthogonal projector, where

$$P_a = \frac{aa^T}{a^Ta}.$$

Can you characterize the subspace onto which P_a is projecting?

Exercise 2 (Least Squares Method)
A planet follows an elliptical orbit that can be described by:

$$ay^2 + bxy + cx + dy + e = x^2, \quad a, b, c, d, e \in \mathbb{R}. \tag{1}$$

(a) Find the quadratic form that best approximates the movement of a planet in the sense of least squares (via normal equations) when the following observations are given:

`x=[1.02,0.95,0.87,0.77,0.67,0.56,0.44,0.30,0.16,0.01];`

`y=[0.39,0.32,0.27,0.22,0.18,0.15,0.13,0.12,0.13,0.15];`

Plot the given data points and the resulting orbit (for example by using a `contour`-plot).

(b) Why is this problem nearly rank deficient? To see the effect of this in the least squares approximation perturb the coordinates of each data point by a random number uniformly distributed on `[-0.005,0.005]`. Plot the new orbit and describe what you observe.

Exercise 3 (Gram-Schmidt Orthogonalization)

(a) Orthogonalize the basis $\{1, x, x^2\}$ of $\mathbb{P}_2 = \{p \text{ polynomial} : \deg(p) \leq 2\}$ via Gram-Schmidt orthogonalization. Use the standard scalar product on $C[-1, 1]$:

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

Have you already seen the resulting polynomials in another context?

(b) Rewrite this result as a QR-decomposition of the "matrix" $A = \left(1 \,\middle|\, x \,\middle|\, x^2\right)$.