

The exercises are to be handed in on Tuesday, 10.01.2012, before the lecture.

Exercise 1 (LU Factorization)

Consider the matrix $A \in \mathbb{R}^{n \times n}$,

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ -1 & 1 & \ddots & \vdots & 1 \\ -1 & -1 & \ddots & 0 & \vdots \\ \vdots & & \ddots & 1 & 1 \\ -1 & -1 & \cdots & -1 & 1 \end{pmatrix}.$$

Make the following Matlab experiment:

- Show that A has a LU-factorization by computing the factor U by hand for arbitrary n .
- Prove: $\|L\|_\infty \leq n$ and $\|U\|_\infty = 2^{n-1}$.
- Choose $n = 100$, $x = \text{ones}(n, 1)$ and $b = Ax$. Solve the system $A\tilde{x} = b$. Compare x and \tilde{x} and identify the problem.
- Perturb A a little by choosing $A_p = A + \text{rand}(n) * 1e - 9$ and carry out step (c) again with the perturbed matrix. What is your observation?

Exercise 2 (Gauss-Seidel Iteration)

Following the steps for the forward Gauss-Seidel iteration presented in the lecture derive the so-called backward Gauss-Seidel iteration by choosing $M = D + U$, $N = -L$ (D diagonal, L lower triangular with zeros on the diagonal, U upper triangular with zeros on the diagonal).

Exercise 3 (Iterative Methods - Comparison)

Let us consider a linear system $Ax = b$, where b is chosen in such a way that the solution is the unit vector $(1, 1, \dots, 1)^T$ and A is the 100×100 tridiagonal matrix whose diagonal entries are all equal to 3, the entries of the first lower diagonal are equal to -2 and those of the upper diagonal are all equal to -1 . Matlab code:

```
n = 100;
A = 4*eye(n) - 2*diag(ones(n-1,1),-1) - diag(ones(n-1,1),1);
b = A*ones(n,1);
```

Compare the number of iteration steps required for Richardson, Jacobi, forward and backward Gauss-Seidel iterations to get the approximation error less than $tol = 10^{-12}$. Plot the error of each method over the number of steps (choose a reasonable scaling of the axes!).

Compare the convergence rate with $\|M^{-1}N\|$.

(The Richardson iteration results from the splitting $A = M - N$, $M = I_n$, $N = I_n - A$.)