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Exercises 1 and 3(b) are to be handed in on Tuesday, 20.12.2011, before the lecture.

Exercise 1 (Gauß Quadrature)

Given $y_0, y_1 \in \mathbb{R}$, explain why all polynomials p of degree ≤ 3 fulfilling

$$p\left(-\frac{1}{\sqrt{2}}\right) = y_0 \quad , \quad p\left(\frac{1}{\sqrt{2}}\right) = y_1$$

give the same value for the integral

$$\int_{-1}^{1} p(x) \frac{1}{\sqrt{1-x^2}} dx \, .$$

Calculate this value in dependence of y_0 and y_1.

Exercise 2 (Monte Carlo Quadrature)

Use the Monte Carlo quadrature method to approximate the integral $\int_2^4 e^x \, dx$ using N samples:
For each $N = 2^k$, $k = 0, \dots, 20$, make 20 independent runs, calculate the mean value, the standard deviation and plot the integration error (you may use the exact value for this) and the standard deviation versus N (loglog-plot).
Estimate the convergence rate.

Exercise 3 (Volume of the Unit Ball)

(a) Let $B_n = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$ be the n-dimensional unit ball.
Write a Matlab program `ball(n,N)` which does the following:

  (i) Use the Matlab function `rand` to sample N uniformly distributed "random" points of the unit cube $W_n = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1\}$ several times ($M = 20$ independent runs)

  (ii) For each run, the percentage a of the points lying in B_n is determined and $v = 2^n a$ is calculated. The output of the program is v, averaged over the 20 independent runs, together with an estimator for the standard deviation.

(b) Relate the output of your program to the volume of the unit ball $Vol(B_n) = \int_{B_n} 1 \, dx$ via Monte Carlo quadrature.

(c) Write another Matlab program which (for fixed dimension n) calculates the error $|Vol(B_n) - v|$ and the (estimated) standard deviation of v and plot these over N (loglog-plot), where $N = 2^k$, $k = 0, \dots, 20$. Test your program for $n = 2,\ 4,\ 10$ und estimate the convergence rate.
For the exact volume of the unit ball use the formula $Vol(B_n) = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}$, where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the Γ-function (Matlab: `gamma(x)`).