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Exercises 1 and 2 are to be handed in on Tuesday, 08.11.2011, before the lecture.

Exercise 1 (Matrix Norm) (∗)

(a) Let $\|.\|$ be a norm on $\mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Prove that the following two definitions for the associated matrix norm are equivalent:

$$\|A\| := \max_{\substack{v\in\mathbb{R}^n,\\ v\neq 0}} \frac{\|Av\|}{\|v\|} \quad \textit{and} \quad \|A\| := \max_{\substack{v\in\mathbb{R}^n,\\ \|v\|=1}} \|Av\|.$$

(b) Consider the maximum norm on $\mathbb{R}^n$ defined by $\|v\|_\infty := \max_{i=1,\dots,n} |v_i|$. Using only the definition in (a), prove that the associated matrix norm is given by $\|A\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^n |a_{ij}|$ (where $A = (a_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$ is an $n \times n$-matrix).
Why is it called the maximum row sum norm?

Hint: Prove that $\|A\| \geq \sum_{j=1}^n |a_{ij}|$ for each i by finding suitable vectors $v_i \in \mathbb{R}^n$ and that $\|A\| \leq \max_{i=1,\dots,n} \sum_{j=1}^n |a_{ij}|$.

Exercise 2 (Quadratic Equations) (∗)

Consider the quadratic equation $a_0 + a_1 x + a_2 x^2 = 0,\ a_0 \neq 0,\ a_2 \neq 0$. Show that

$$x_1 = \frac{+\sqrt{a_1^2 - 4a_0a_2} - a_1}{2a_2}, \quad x_2 = \frac{-\sqrt{a_1^2 - 4a_0a_2} - a_1}{2a_2}$$

solve the upper equation. Prove that these solutions can also be written as

$$x_1 = \frac{2a_0}{-\sqrt{a_1^2 - 4a_0a_2} - a_1}, \quad x_2 = \frac{2a_0}{+\sqrt{a_1^2 - 4a_0a_2} - a_1}.$$

Which of these formulas would you use to compute x_1, which one to compute x_2?

Hint: Examine the cases $a_1 \geq 0$, $a_1 < 0$ separately and recall that cancellation occurs whenever two similar values are subtracted from one another (consider the cases where $|a_0|$ or $|a_2|$ are very small).

Exercise 3 (Wilkinson Polynomial)
Consider the Wilkinson polynomial $w(x) := \prod_{k=1}^{20}(x - j) = a_{20}x^{20} + \cdots + a_1 x + a_0$, which obviously has the roots $x_j = j,\ j = 1, \ldots, 20$. You can find the (approximative) coefficients $a_{20}, \ldots, a_0$ by using the Matlab function $poly(r)$ where $r = [1:20]$ is the vector of the roots. Now perturb the coefficients and plot the new (complex) roots in the complex plane:

1. do not perturb at all,
2. multiply a_0 by $(1 + 10^{-5})$,
3. multiply a_{10} by $(1 + 10^{-5})$,
4. multiply a_{15} by $(1 + 10^{-5})$.

You can use the Matlab-function $roots(p)$, where p is the vector $[a_{20}, \ldots, a_0]$. Please pay attention to the order of the coefficients a_i. Use different colours for the plots of (a),(b),(c),(d).