Caroline Lasser, Ilja Klebanov

The exercises are to be handed in on Tuesday, 24.01.2012, before the lecture.

Exercise 1 (Householder Reflector)

Let $v \in \mathbb{R}^n$. Find the eigenvalues and eigenvectors of the Householder reflector

$$F = I - 2\frac{vv^T}{v^Tv}.$$

Give a geometric argument for your observation.
Hint: Decompose a vector $x \in \mathbb{R}^n$ into a part parallel to v and a part perpendicular to v: $x = \alpha v + w,\ w \perp v$. Draw a picture!

Exercise 2 (Householder Reflector)

Show that for any two vectors $x, y \in \mathbb{R}^n$ of the same length ($\|x\|_2 = \|y\|_2$) the choice $v = x - y$ leads to a Householder transformation $F = I - 2\frac{vv^T}{v^Tv}$ such that $Fx = y$ and $Fy = x$. Give a geometric argument for this.
Hint: It might be helpful to first consider the vector $z = \frac{x+y}{2}$ and apply F to it. Draw a picture!

Exercise 3 (Givens Rotation)

Let $\theta \in [0, 2\pi)$ and $c = \cos(\theta),\ s = \sin(\theta)$. Consider matrices $Q \in \mathbb{R}^{3\times 3}$ of the form

$$Q_1 = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad Q_2 = \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix} \quad \text{or} \quad Q_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}.$$

(a) Show that the Q_i are orthogonal matrices (it is sufficient if you show it for one of them). Explain geometrically what the matrices do when applied to a vector $x \in \mathbb{R}^3$.

(b) Given the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \\ -2 & 3 \end{pmatrix},$$

Choose an angle $\theta \in [0, 2\pi)$ and an index $i \in \{1, 2, 3\}$, such that the product Q_iA is of the form

$$Q_iA = \begin{pmatrix} \times & \times \\ \times & \times \\ 0 & \times \end{pmatrix},$$

i.e. Q_i eliminates the left bottom entry of A.

Hint: $c, s \in \mathbb{R}$ fulfill $c = \cos(\theta),\ s = \sin(\theta)$ for some θ if and only if $c^2 + s^2 = 1$.