Gaussian Beam Approximations

Olof Runborg

CSC, KTH

Joint with Hailiang Liu, Iowa, and Nick Tanushev, Austin

Technischen Universität München
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High frequency waves

Cauchy problem for scalar wave equation

\[ u_{tt} - c(x)^2 \Delta u = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \]

\[ u(0, x) = A(x)e^{i\phi(x)}/\varepsilon, \quad u_t(0, x) = \frac{1}{\varepsilon} B(x)e^{i\phi(x)}/\varepsilon, \]

where \( c(x) \) (variable) speed of propagation.
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- High frequency → short wave length → highly oscillatory solutions → many gridpoints.

Solution \( u(x,y) \)
High frequency waves

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- Direct numerical solution resolves wavelength:
  \[ \#\text{gridpoints} \sim \varepsilon^{-n} \text{ at least} \Rightarrow \text{cost} \sim \varepsilon^{-n-1} \text{ at least} \]
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  \#gridpoints \sim \varepsilon^{-n} \text{ at least} \Rightarrow \text{cost} \sim \varepsilon^{-n-1} \text{ at least}
- Often unrealistic approach for applications in e.g. optics, electromagnetics, geophysics, acoustics, ...
Geometrical optics

Wave equation

\[ u_{tt} - c(x)^2 \Delta u = 0. \]

Write solution on the form

\[ u(t, x) = a(t, x, \varepsilon)e^{i\phi(t,x)/\varepsilon}. \]
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(a) Amplitude \( a(x) \)

(b) Phase \( \phi(x) \)

Solution \( u(x, y) \)
Geometrical optics

- $a, \phi$ vary on a much coarser scale than $u$. (And varies little with $\varepsilon$.) Geometrical optics approximation considers $a$ and $\phi$ as $\varepsilon \to 0$. 

\[
\begin{align*}
\phi^2_t - c(y)^2|\nabla \phi|^2 &= 0, \\
\alpha_t + c \nabla \phi \cdot \nabla \alpha |\nabla \phi|^2 + c^2 \Delta \phi - \phi_{tt} 2c |\nabla \phi|^2 \alpha &= 0,
\end{align*}
\]

Good accuracy for small $\varepsilon$. Computational cost $\varepsilon$-independent.

\[
u(t, x) = a(t, x) e^{i \phi(t, x) / \varepsilon} + O(\varepsilon)\]

Waves propagate as rays, c.f. visible light. Not all wave effects captured correctly.
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- Phase and amplitude satisfy eikonal and transport equations

\[
\phi_t^2 - c(y)^2 |\nabla \phi|^2 = 0, \quad a_t + c \frac{\nabla \phi \cdot \nabla a}{|\nabla \phi|} + \frac{c^2 \Delta \phi - \phi_{tt}}{2c|\nabla \phi|} a = 0.
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Numerical approaches

- Eikonal and transport equation (PDEs)

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Raytracing (ODEs)

Rays are the (bi)characteristics $\left( x(t), p(t) \right)$ of the eikonal equation,

$$\frac{dx}{dt} = c(x)^2 p, \quad \frac{dp}{dt} = -\frac{\nabla c(x)}{c(x)}, \quad \phi(t, x(t)) = \phi(0, x(0)).$$

$p(t)$ is local ray direction, "slowness" vector ($|p| = 1/c$ and $p = \nabla \phi(x)$). (Also ODEs for amplitude.)
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Difficulties:
- Nonlinearity of eikonal equation, viscosity solution, kinks
- Multiphase solutions, crossing rays
- Diverging rays
- Breakdown of geometrical optics (boundaries, caustics)
Caustics

Concentration of rays.

Amplitude $a(t, y) \to \infty$ but should be $a(t, y) \sim \varepsilon^{-\alpha}$, $0 < \alpha < 1$. 
Schrödinger Case

Same conclusions also for the time-dependent Schrödinger equation

\[ i\varepsilon u_t + \varepsilon^2 \Delta u - V(x)u = 0. \]

Geometrical optics with

\[ u(t, x) = a(t, x)e^{i\phi(t, x)/\varepsilon} + O(\varepsilon), \]

- Eikonal and transport equation

\[ \phi_t + |\nabla \phi|^2 - V(x)\phi = 0, \quad a_t + 2\nabla \phi \cdot \nabla a + a\Delta \phi = 0. \]
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\[ \frac{dx}{dt} = p, \quad \frac{dp}{dt} = -\nabla V(x), \quad \frac{d\phi}{dt} = \frac{1}{2} |p|^2 - V(x). \]
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Outline

Improved high frequency asymptotic approximations based on Gaussian beams

1. Construction and derivation
2. Error estimates in terms of $\varepsilon$
3. Numerics (some)
Constructing asymptotic solutions

1. Make an ansatz for the form the solution $\tilde{u}(t, y)$ with some unknown coefficients/functions, e.g.

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\[
\tilde{u}(t, y) = a(t, y)e^{i\Phi(t,y)/\varepsilon},
\]

2. Find the coefficients/functions in the ansatz such that \( \tilde{u} \) satisfies the Schrödinger equation as well as possible,

\[
\left| i\varepsilon \tilde{u}_t + \varepsilon^2 \Delta \tilde{u} - V(x)\tilde{u} \right| \ll 1.
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3. ... and such that the coefficients can be numerically computed much easier than the full solution $u$ itself.
Use WKB expansion as ansatz \((a(t, y) \text{ a power series in } i\varepsilon)\)

\[
    u(t, y) = e^{i\Phi(t, y)/\varepsilon} \sum_{k=0}^{K-1} a_k(t, y)(i\varepsilon)^k.
\]

Find the coefficients in the ansatz.

\[
    i\varepsilon u_t + \varepsilon^2 \Delta u - V u = -E[\Phi] u + i\varepsilon P[a_0] e^{i\Phi/\varepsilon} + K^{-1} \sum_{k=0}^{K-2} (i\varepsilon)^k + 2 \left( P[a_{k+1}] - \Delta a_k \right) e^{i\Phi/\varepsilon} - (i\varepsilon)K^{-1} \Delta a_K e^{i\Phi/\varepsilon}
\]

where

\[
    E[\Phi] := \Phi_t + |\nabla \Phi|^2 + V \Phi,
\]

\[
    P[a] := a_t + 2 \nabla \phi \cdot \nabla a + a \Delta \phi.
\]

Solve \(E[\Phi] = 0, P[a_0] = 0\) and \(P[a_{k+1}] = \Delta a_k\).

Non-oscillatory problems, easier to solve.
Example: Geometrical optics

1. Use WKB expansion as ansatz \((a(t, y) \text{ a power series in } i\varepsilon)\)

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2. Find the coefficients in the ansatz.

\[ i\varepsilon u_t + \varepsilon^2 \Delta u - Vu = -\varepsilon[\Phi]u + i\varepsilon\mathcal{P}[a_0]e^{i\Phi/\varepsilon} \]

\[ + \sum_{k=0}^{K-2} (i\varepsilon)^{k+2} (\mathcal{P}[a_{k+1}] - \Delta a_k) e^{i\Phi/\varepsilon} - (i\varepsilon)^{K+1} \Delta a_{K-1} e^{i\Phi/\varepsilon} \]

where \(\varepsilon\) and \(\mathcal{P}\) are the eikonal and transport operators

\[ \varepsilon[\Phi] := \Phi_t + |\nabla \Phi|^2 + V\Phi, \quad \mathcal{P}[a] := a_t + 2\nabla \phi \cdot \nabla a + a \Delta \phi. \]
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\[ \mathcal{E}[\Phi] := \Phi_t + |\nabla \Phi|^2 + V \Phi, \quad \mathcal{P}[a] := a_t + 2 \nabla \phi \cdot \nabla a + a \Delta \phi. \]

Solve \(\mathcal{E}[\Phi] = 0, \mathcal{P}[a_0] = 0\) and \(\mathcal{P}[a_{k+1}] = \Delta a_k\). Then,

\[ i\epsilon u_t + \epsilon^2 \Delta u - Vu = O(\epsilon^{K+1}) \]
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3. Non-oscillatory problems, easier to solve.
Gaussian approximations

Approximate solutions to the wave equations/Schrödinger with a Gaussian profile (width $\sim \sqrt{\varepsilon}$).
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- For waves studied by e.g. Cerveny, Popov, Babich, Psencik, Ralston, Hörmander, Klimes, Hill, etc.
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- For Schrödinger, classical (coherent state), Heller, Hagedorn, Herman, Kluk, Kay, etc.
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- For Schrödinger, classical (coherent state), Heller, Hagedorn, Herman, Kluk, Kay, etc.
- No break down at caustics. Gives e.g. improved seismic imaging.
Approximation of the same form as geometrical optics solutions,

\[ v(t, y) = a(t, y)e^{i\Phi(t, y)/\varepsilon}, \quad \Phi(t, y) = \phi(t, y - x(t)), \]

where \( x(t) \) is a geometrical optics ray.
Gaussian approximations

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\[ \nu(t, y) = a(t, y)e^{i\Phi(t, y)/\varepsilon}, \quad \Phi(t, y) = \phi(t, y - x(t)), \]

where \( x(t) \) is a geometrical optics ray.

- The phase \( \Phi \) will now have a positive imaginary part away from the ray \( x(t) \).

Gaussian Beam Approximations
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  - Gaussian with width \( \sqrt{\varepsilon} \)
  - Localized around \( x(t) \). (Moves along the space time ray.)
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- Gaussian with width \( \sqrt{\varepsilon} \)
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Amplitude \( a(t, y) \) and the phase \( \Phi(t, y) \) approximated by polynomials locally around \( x(t) \).
The simplest ("first order") Gaussian beams use the ansatz

\[ v(t, y) = a(t) e^{i\Phi(t, y)/\varepsilon}, \quad \Phi(t, y) = \phi(t, y - x(t)), \]

where

\[ \phi(t, y) = \phi_0(t) + y \cdot p(t) + \frac{1}{2} y \cdot M(t) y. \]

i.e. \( a(t, y) \) approximated to 0th order, and \( \Phi(t, y) \) to 2nd order.
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⇒

We can require that \(\Phi(t, y)\) solves eikonal to order \(O(|y - x|^3)\) and \(a(t, y)\) solves transport equation to order \(O(|y - x|)\).
Let us thus require that

\[ \Phi_t + |\nabla \Phi|^2 - V(y)\Phi = O(|y - x(t)|^3), \quad a_t + 2\nabla \Phi \cdot \nabla a + a\Delta \Phi = O(|y - x(t)|). \]
First order beams

Let us thus require that

\[ \dot{\Phi} + |\nabla \Phi|^2 - V(y) \Phi = O(|y-x(t)|^3), \quad a_t + 2\nabla \Phi \cdot \nabla a + a \Delta \Phi = O(|y-x(t)|). \]

Then we obtain ODEs for \( \phi_0, x, p, M, a_0 \).

\[
\begin{align*}
\dot{x}(t) &= p, \\
\dot{p}(t) &= -\nabla V(x), \\
\dot{\phi}_0(t) &= \frac{1}{2} |p|^2 - V(x), \\
\dot{M}(t) &= -M^2 - \partial_x^2 V(x), \\
\dot{a}_0(t) &= -\frac{1}{2} a_0 \text{Tr}(M).
\end{align*}
\]

Easy to compute numerically!
Asymptotic order

As before,

\[ i\varepsilon v_t + \varepsilon^2 \Delta v - Vv = -\varepsilon[\Phi]u + i\varepsilon \mathcal{P}[a]e^{i\Phi/\varepsilon} - (i\varepsilon)^2 \Delta ae^{i\Phi/\varepsilon} \]
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Here we enforce

\[ \varepsilon[\Phi] = O(|y - x|^3), \quad \mathcal{P}[a] = O(|y - x|), \]

and \( \Delta a = \Delta a_0(t) = 0 \) by construction.
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Hence, since \( |\exp(i\Phi/\varepsilon)| \sim \exp(-|y - x|^2/\varepsilon) \),

\[ i \varepsilon v_t + \varepsilon^2 \Delta v - Vv = O \left( \left| |y - x|^3 + \varepsilon|y - x| \right| e^{-\frac{|y-x|^2}{\varepsilon}} \right). \]
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Using \( x^p e^{-x^2/\varepsilon} \leq C_p \varepsilon^{p/2} \) we then get

\[ i\varepsilon v_t + \varepsilon^2 \Delta v - Vv = O \left( \varepsilon^{3/2} \right) \]
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\[ i\varepsilon v_t + \varepsilon^2 \Delta v - Vv = -\varepsilon[\Phi]u + i\varepsilon\mathcal{P}[a]e^{i\Phi}/\varepsilon - (i\varepsilon)^2 \Delta a e^{i\Phi}/\varepsilon \]

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(C.f. GO: \( i\varepsilon u_t + \varepsilon^2 \Delta u - Vu = O(\varepsilon^2) \).)
Gaussian beams

Properties

\[ v(t, y) = a_0(t) e^{i \phi(t, y-x(t))/\varepsilon}, \quad \phi(t, y) = \phi_0(t) + y \cdot p(t) + \frac{1}{2} y \cdot M(t)y \]

- \( \Phi(t, x(t)) = \phi(t, 0) = \phi_0(t) \) is real valued
- If \( M(0) \) is symmetric and \( \Im M(0) \) is positive definite then this is true for \( M(t) \) (which exists) for all \( t > 0 \).
- Second derivatives of \( \phi \) exist everywhere (no blow-up at caustics)
- Shape of beam remains Gaussian
Higher order beams

More generally, we can construct higher order beams. Let

$$v(t, y) = a(t, y - x(t)) e^{i \phi(t, y - x(t))/\epsilon},$$

where, for order $K$ beams,
Higher order beams

More generally, we can construct higher order beams. Let

$$v(t, y) = a(t, y - x(t))e^{i\phi(t, y - x(t))/\varepsilon},$$

where, for order $K$ beams,

- The phase is a Taylor polynomial to order $K + 1$,

$$\phi(t, y) = \phi_0(t) + y \cdot p(t) + y \cdot \frac{1}{2} M(t)y + \sum_{|\beta| = 3}^{K+1} \frac{1}{\beta!} \phi_\beta(t)y^\beta.$$
More generally, we can construct higher order beams. Let
\[ v(t, y) = a(t, y - x(t)) e^{i\phi(t, y - x(t))/\varepsilon}, \]
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- Each amplitude term \( a_j \) is a Taylor polynomial to order \( K - 2j - 1 \)

\[ a_j(t, y) = \sum_{|\beta|=0}^{K-2j-1} \frac{1}{\beta!} a_{j,\beta}(t)y^\beta \]
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We now require that

- $\phi(t, y)$ solves the eikonal equation to order $|y - x|^{K+2}$
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This gives ODEs for all Taylor coefficients,

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\begin{align*}
\dot{x}(t) &= p, \\
\dot{p}(t) &= -\nabla V(x), \\
\dot{\phi}_0(t) &= \frac{1}{2} |p|^2 - V(x), \\
\dot{M}(t) &= -M^2 - \partial_x^2 V(x), \\
\dot{a}_{j,\beta}(t) &= \ldots, \\
\dot{\phi}_\beta(t) &= \ldots,
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\end{align*}
\]

Again, \( \Phi(t, x(t)) \) is real and \( \Im M(t) > 0 \Rightarrow \) no problems at caustics.
Asymptotic order

As before, with $\tilde{K} = \lceil K/2 \rceil$,

$$i\varepsilon v_t + \varepsilon^2 \Delta v - Vv = -\varepsilon [\Phi] v + i\varepsilon P[a_0] e^{i\Phi/\varepsilon}$$

$$+ \sum_{k=0}^{\tilde{K}-2} (i\varepsilon)^{k+2} (P[a_{k+1}] - \Delta a_k) e^{i\Phi/\varepsilon} - (i\varepsilon)^{\tilde{K}+1} \Delta a_{\tilde{K}-1} e^{i\Phi/\varepsilon}$$
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Here we enforce that

$$\mathcal{E}[\Phi] = O(|y-x|^{K+2}), \quad \mathcal{P}[a_0] = O(|y-x|^K),$$

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Recalling that \( x^p e^{-x^2/\varepsilon} \leq C_p \varepsilon^{p/2} \),

\[
i \varepsilon v_t + \varepsilon^2 \Delta v - Vv = O \left( \varepsilon^{K/2+1} \right)
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Thawed Gaussian Approximation [Heller, 75]
Let $\phi$ always be a second order polynomial. Take higher order polynomial in amplitude. (Use more terms in amplitude to correct also for errors in phase.)
Other gaussian beam like approximations

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- Frozen Gaussian Approximation [Heller, 81], [Herman, Kluk, 84]
  Let $\phi$ always be a second order polynomial, with a fixed second derivative ($M(t) = \text{constant}$). Single frozen Gaussian not an asymptotic solution. Use superposition of frozen Gaussians. ($\sim$ linear basis expansion)
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- Gives ODEs for coefficients.
Approximation Errors

Suppose $u$ is exact solution

$$i\varepsilon \partial_t u + \varepsilon^2 \Delta u - Vu = 0.$$  

and $\tilde{u}$ is the approximate asymptotic solution,

$$i\varepsilon \partial_t \tilde{u} + \varepsilon^2 \Delta \tilde{u} - V\tilde{u} \ll 1.$$  

What is the norm error in $\tilde{u}$, i.e. $||u - \tilde{u}||$?
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What is the norm error in \( \tilde{u} \), i.e. \( \| u - \tilde{u} \| \)?

Use well-posedness (stability) estimate for PDE:

\[
i \varepsilon \partial_t w + \varepsilon^2 \Delta w - Vw = f(t, x),
\]

implies that for \( 0 \leq t \leq T \),

\[
\| w(t, \cdot) \|_{L^2} \leq \| w(0, \cdot) \|_{L^2} + \frac{C(T)}{\varepsilon} \sup_{t \in [0, T]} \| f(t, \cdot) \|_{L^2}.
\]
Define $P^\varepsilon$ as

$$P^\varepsilon[w] := i\varepsilon \partial_t w + \varepsilon^2 \Delta w - Vw.$$ 

Then $P^\varepsilon[u] = 0$ and

$$P^\varepsilon[\tilde{u} - u] = P^\varepsilon[\tilde{u}] - P^\varepsilon[u] = P^\varepsilon[\tilde{u}].$$

Moreover, the well-posedness estimate can be rewritten

$$\|w(t, \cdot)\|_{L^2} \leq \|w(0, x)\|_{L^2} + \frac{C(T)}{\varepsilon} \sup_{t \in [0, T]} \|P^\varepsilon[w](0, \cdot)\|_{L^2}.$$
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Hence, assuming $u(0, x) = \tilde{u}(0, x),$}

$$\|\tilde{u}(t, \cdot) - u(t, \cdot)\|_{L^2} \leq \frac{C(T)}{\varepsilon} \sup_{t \in [0,T]} \|P^\varepsilon[\tilde{u}](t, \cdot)\|_{L^2}, \quad 0 \leq t \leq T.$$ 

Error in $\tilde{u} \sim$ how well it satisfies equation, minus one order in $\varepsilon$. 
If \( \phi \) and \( A \) satisfy eikonal equation and (high order) transport equations, then for

\[
\tilde{u}_{GO} = A(t, x)e^{i\phi(t,x)/\varepsilon},
\]

we have

\[
P^\varepsilon [\tilde{u}_{GO}] (t, x) = O(\varepsilon^{K+1}),
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before caustics develop.
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before caustics develop.

Hence,
\[ \| u - \tilde{u}_{GO} \|_{L^2} \leq \frac{C}{\varepsilon} \| P^\varepsilon [\tilde{u}_{GO}] \|_{L^2} \sim O(\varepsilon^K). \]
(As expected since we cut off WKB expansion at $K - 1$.)
Example 2: Single Gaussian Beam

By earlier construction

\[ P^\varepsilon [\tilde{u}_{GB}] (t, x) = O(\varepsilon^{K/2+1}). \]

Note here that since width of beam \( \sim \sqrt{\varepsilon}, \)

\[ \| \tilde{u}_{GB} \|_{L^2} \sim \varepsilon^{n/4} \]

and

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Therefore, relative error is

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Proofs of this e.g. by [Lax, 57] for geometrical optics, [Ralston, 82] for Gaussian beams in wave setting and [Hagerdorn, 85] for Thawed Gaussians.
Superpositions of Gaussian beams

To approximate more general solutions, use superpositions of beams. Let \( v(t, y; z) \) be a beam starting from the point \( y = z \) and define

\[
u_{GB}(t, y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t, y; z) dz.
\]

\((n - \text{dimension}, K_0 - \text{compact set})\)
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Sufficient to describe e.g. WKB data \( a(y)e^{i\phi(y)}/\varepsilon \).
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Sufficient to describe e.g. WKB data $a(y)e^{i\phi(y)/\varepsilon}$.

- Prefactor scales beams appropriately ($u_{GB} = O(1)$ if $v = O(1)$).
More general phase space superposition:

Let $v(t, y; z, p)$ be a beam starting from the point $y = z$ with momentum $p$ and define

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Can describe much more general data. (C.f. FBI transform.)
Numerical methods

- Approximate superposition integral by sum (trapezoidal rule)

\[ u_{GB}(t, y) = \varepsilon^{-\frac{n}{2}} \int_{K_0} v(t, y; z) dz \approx \varepsilon^{-\frac{n}{2}} \sum_{j} v(t, y; z_j) \Delta z^n. \]
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Lagrangian methods – Solve ODEs with standard methods. Similar to ray tracing but with all the additional Taylor coefficients computed along the rays \((M, a_j, \beta, \phi_\beta, \ldots)\) [Hill, Klimes, \ldots]
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Wavefront methods – solve for parameters on a wave front [Motamed, OR,09]
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**Numerical issues**

- Cost \( \sim \) number of beams since each beam is \( O(1) \).
  - For accuracy need \( \Delta z \sim \sqrt{\varepsilon} \sim \) width of beams.
  - \( \Rightarrow \) cost \( \sim O(\varepsilon^{-n/2}) \)
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- Spreading of beams
  - Wide beams \( \Rightarrow \) large Taylor approximation errors
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- Initial data approximation
  Many degrees of freedom. Can have huge impact on accuracy at later times.
Approximation Errors Superpositions

Norm estimates of $\|u - u_{GB}\|$ only rather recently derived
[Swart, Rousse, Liu, Ralston, Tanushev, Bougacha, Alexandre, ...]

Need to check how well $u_{GB}$ satisfies equation

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- Coarse estimate gives

$$||P^{\varepsilon}[u_{GB}]|| \leq \varepsilon^{-n/2} \int_{K_0} ||P^{\varepsilon}[v(t, y; z)]|| dz \leq C\varepsilon^{-n/2} \varepsilon^{K/2 + 1 + n/4}.$$
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- By linearity,
  $$P^\varepsilon[u_{GB}] = \varepsilon^{-n/2} \int_{K_0} P^\varepsilon[v(t, y; z)] dz.$$  

- Coarse estimate gives
  $$\|P^\varepsilon[u_{GB}]\| \leq \varepsilon^{-n/2} \int_{K_0} \|P^\varepsilon[v(t, y; z)]\| dz \leq C\varepsilon^{-n/2} \varepsilon^{K/2 + 1 + n/4}.$$  

- Gives error estimate for $u_{GB}$
  $$\|u - u_{GB}\| \leq \frac{C}{\varepsilon} \|P^\varepsilon[u_{GB}]\| \leq C\varepsilon^{K/2 - n/4}.$$
Basic estimate for the Schrödinger equation, [Liu, Ralston, 2010],

\[ \| u(t, \cdot) - u_{GB}(t, \cdot) \|_{L^2} \leq O(\varepsilon^{K/2-n/4}) . \]

is not sharp. E.g. it does not predict convergence for first order beams in 2D.
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- Problem: The estimate

\[ |P^\varepsilon[u_{GB}]| \leq \varepsilon^{-n/2} \int_{K_0} |P^\varepsilon[v(t, y; z)]|dz \]

ignores cancellations between neighbouring beams. Very bad except at caustics where beams interfere constructively.
Approximation Errors Superpositions

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For phase space superposition, results with no dimensional dependence:

For the wave equation with phase space superposition,

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Main result

Theorem

For scalar, strictly hyperbolic $m$-th order PDEs

$$
\varepsilon^{m-1} \left( \sum_{\ell=0}^{m-1} \| \partial^\ell_t [u(t, \cdot) - u_{GB}(t, \cdot)] \|_{H^{m-\ell-1}}^2 \right)^{1/2} \leq O(\varepsilon^{K/2}).
$$

For the wave equation,

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\| u(t, \cdot) - u_{GB}(t, \cdot) \|_E \leq O(\varepsilon^{K/2}).
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- Superposition in physical space. Initial data approximated on a submanifold of phase space (WKB data).
- Convergence of all beams independent of dimension and presence of caustics.
Sketch of proof

For all the PDEs $P[u] = 0$ considered we have an energy estimate of the type

$$\|u_{GB}(t, \cdot) - u(t, \cdot)\|_S \leq \|u_{GB}(0, \cdot) - u(0, \cdot)\|_S + C\varepsilon^q \int_0^t \|P[u_{GB}](\tau, \cdot)\|_{L^2} d\tau,$$

where $\|\cdot\|_S$ is some appropriate norm and $q$ is an integer. (E.g. Schrodinger $S = L^2, q = -1$, wave equation $S = E, q = 1$.)
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- For all PDEs considered, we can write

$$
P[u_{GB}](t, y) = \varepsilon^{K/2-q} \sum_{j=1}^J \varepsilon^{r_j} T_j^{\varepsilon}[f_j](t, y) + O(\varepsilon^\infty),
$$

where $r_j \geq 0$, $J$ finite and $f_j \in L^2$ (all independent of $\varepsilon$).

$T_j^{\varepsilon} : L^2 \rightarrow L^2$ belongs to a class of oscillatory integral operators.
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Together we get (if initial data exact)

$$\|u_{GB}(t, \cdot) - u(t, \cdot)\|_S \leq C(T) \varepsilon^{K/2} \sum_{j=1}^J \varepsilon^{r_j} \|T_j^\varepsilon\|_{L^2} \|f_j\|_{L^2} + O(\varepsilon^\infty).$$
Sketch of proof, cont.

We have

\[ \| u_{GB}(t, \cdot) - u(t, \cdot) \|_S \leq C(T)\varepsilon^{K/2} \sum_{j=1}^{J} \| T_j^\varepsilon \|_{L^2} + \mathcal{O}(\varepsilon^{\infty}) \]

where, in its simplest form,

\[ T^\varepsilon [w](t, y) := \varepsilon^{-n+|\alpha|/2} \int_{K_0} w(z)(y - x(t; z))^{\alpha} e^{i\phi(t, y - x(t; z); z)/\varepsilon} \, dz, \]

for some multi-index \( \alpha \), Gaussian beam phase \( \phi \) and geometrical optics rays \( x(t; z) \) with \( x(0; z) = z \).
Sketch of proof, cont.

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Result follows if we prove that \( T^\varepsilon \) is bounded in \( L^2 \) independent of \( \varepsilon \),

\[ \| T^\varepsilon \|_{L^2} \leq C. \]

This is the key estimate of our proof.
Sketch of proof, cont.

Estimate of $||\mathcal{T}^\varepsilon||_{L^2}$, where

$$\mathcal{T}^\varepsilon[w](t, y) := \varepsilon^{-\frac{n+|\alpha|}{2}} \int_{K_0} w(z)(y - x(t; z))^{\alpha} e^{i\phi(t, y - x(t; z); z)/\varepsilon} dz.$$ 

Main difficulty: no globally invertible map $x(0; z) = z \rightarrow x(t; z)$ because of caustics.
Sketch of proof, cont.

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- Mapping $(x(0; z), p(0; z)) \rightarrow (x(t; z), p(t; z))$ is however globally invertible and smooth. Gives the "non-squeezing" property,

$$c_1|z - z'| \leq |p(t; z) - p(t; z')| + |x(t; z) - x(t; z')| \leq c_2|z - z'|.$$  

Allows us to use stationary phase arguments close to caustics, and carefully control cancellations of oscillations there (similar to [Swart,Rousse], [Bougacha, Akian, Alexandre]).
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Approximation errors

Remarks

The estimate

$$\|u(t, \cdot) - u_{GB}(t, \cdot)\|_{E} \leq O(\varepsilon^{K/2})$$

is sharp for individual beams (relative error). But for superpositions?
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- Predicts convergence rate of first order beam to be only \( O(\sqrt{\varepsilon}) \). These beams are based on same high frequency approximation as geometrical optics which has \( O(\varepsilon) \) accuracy.
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- Predicts convergence rate of first order beam to be only $O(\sqrt{\varepsilon})$. These beams are based on same high frequency approximation as geometrical optics which has $O(\varepsilon)$ accuracy.
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for the Taylor expansion part of the error away from caustics. This gives \( O(\varepsilon) \) for first order beams.
- More error cancellations coming in for odd order beams? (\( \Rightarrow \) no gain in using even order beams)
- Numerical experiments also in the time-dependent case suggests a better rate for odd order beams
Numerical examples

Cusp caustic

Consider the test case where

\[ \Phi(0, y) = -y_1 + y_2^2, \]
\[ A(0, y) = e^{-10|y|^2}. \]

- Cusp caustic at \( t = 0.5 \)
- Two fold caustics at \( t > 0.5 \)
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Numerical examples
Cusp caustic, convergence

\begin{align*}
\|u_k(0.25,\cdot)-u_F(0.25,\cdot)\|_E \\
\|u_k(0.75,\cdot)-u_F(0.75,\cdot)\|_E
\end{align*}

\begin{figure}
\centering
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{plot1}
\end{subfigure}
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{plot2}
\end{subfigure}
\end{figure}
Numerical examples

Cusp caustic, convergence
For any $y$ away from caustics we can prove that

$$|u_{GB}(t, y) - u(t, y)| \leq C(y)\varepsilon^{[K/2]}.$$
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1. Prove error estimates in higher order Sobolev spaces

$$\|u_{GB}(t, \cdot) - u(t, \cdot)\|_{H^s} = O(\varepsilon^{K/2-s}).$$
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\|u_{GB}(t, \cdot) - u(t, \cdot)\|_{H^s} = O(\varepsilon^{K/2-s}).
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2. Use Sobolev inequalities and \( K + m \) order beams \( u_{GB}^{K+m} \) to show that
\[
|u_{GB}^{K+m}(t, \cdot) - u(t, \cdot)|_{L^\infty} \leq \varepsilon^{[K/2]}.
\]
for large enough \( m \).
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for large enough $m$.

3. Consider difference between $K$ order and $K + m$ order beams,

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4. Estimate $T_j^{\varepsilon}$ in max norm.