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Übung zu Kreuzprodukten

1. *Kreuzprodukt*
   Seien $a, b \in \mathbb{R}^3$. Zeigen und interpretieren Sie:
   a) $a \times b = -(b \times a) = (-b) \times a = b \times (-a)$,
   b) $a \times a = 0$.

   Man definiert den Winkel $\alpha \in [0, \pi]$ zwischen a und b durch
   $$\cos(\alpha) = \frac{\langle a, b\rangle}{\|a\|\|b\|}.$$
   c) Zeigen Sie mittels Volumenformeln $\langle a, b\rangle^2 + \|a \times b\|^2 = \|a\|^2\|b\|^2$.
   d) Zeigen Sie $\sin(\alpha) = \|a \times b\|/(\|a\|\|b\|)$.

2. *Positive definite matrices*
   A positive definite matrix is a matrix $A \in \mathbb{C}^{n\times n}$ that satisfies
   $$\langle x, Ax\rangle > 0$$
   for all $x \in \mathbb{C}^n \backslash \{0\}$. We denote by $\langle x, y\rangle = x^* y$ the standard scalar product on $\mathbb{C}^n$.
   a) Assume that $A \in \mathbb{C}^{n\times n}$ is hermitian and positive definite. Prove that
   $$\langle x, y\rangle_A = \langle x, Ay\rangle$$
   defines a scalar product on $\mathbb{C}^n$.
   b) Prove for all $A \in \mathbb{C}^{n\times n}$ and $x, y \in \mathbb{C}^n$ the two polarization identities:
   $$4\langle Ax, y\rangle = \sum_{k=0}^{3} (-i)^k \langle A(x + i^k y), x + i^k y\rangle,$$
   $$4\langle x, Ay\rangle = \sum_{k=0}^{3} (-i)^k \langle x + i^k y, A(x + i^k y)\rangle$$