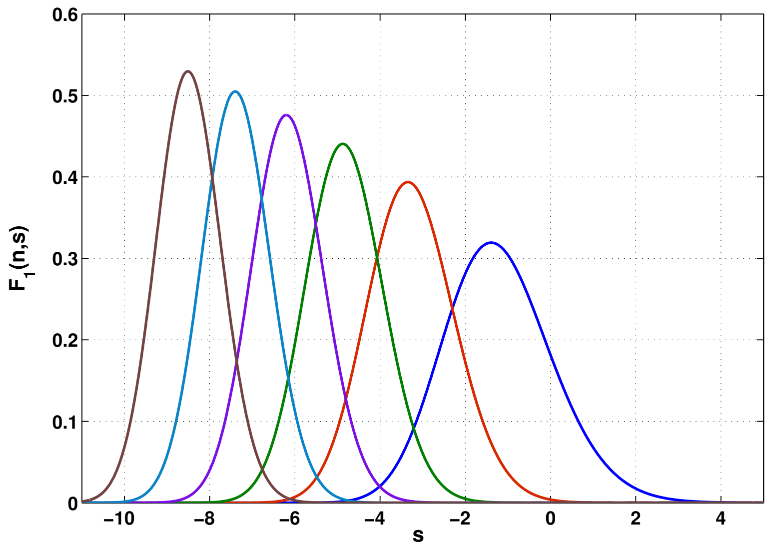
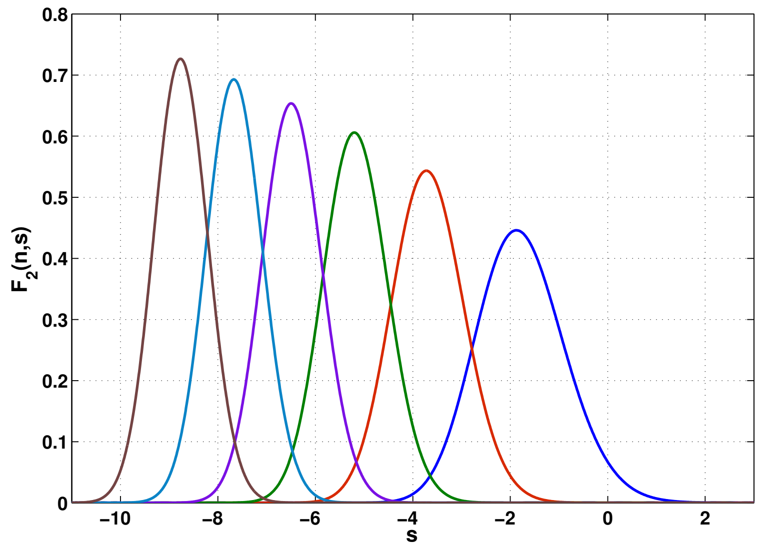
Numerical Evaluation of Distributions in (*Integrable*) Random Matrix Theory



probability density of n-th largest level in edge scaled GOE



probability density of n-th largest level in edge scaled GUE

Folkmar Bornemann

Tools Used for Exact Solutions

Ivar Fredholm (1866–1927)

determinant of integral operator (1899)

$$Ku(x) = \int_a^b K(x,y)u(y)\,dy \qquad \rightsquigarrow$$

$$\det(\mathbb{1} + zK) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \int_{[a,b]^m} \det\left(K(t_i,t_j)\right)_{i,j=1}^{m} dt$$

Paul Painlevé (1863–1933)

six families of irreducible transcendental functions (1895)

$$u_{xx} = 6u^2 + x$$

$$u_{xx} = 2u^3 + xu - \alpha$$

$$u_{xx} = u^{-1}u_x^2 - x^{-1}u_x + x^{-1}(\alpha u^2 + \beta) + \gamma u^3 + \delta u^{-1}$$

$$u_{xx} = (2u)^{-1}u_x^2 + 3u^3/2 + 4xu^2 + 2(x^2 - \alpha)u + \beta u^{-1}$$

$$u_{xx} = (3u-1)(2u(u-1))^{-1}u_x^2 - x^{-1}u_x + \gamma x^{-1}u + (u-1)^2 x^{-2}(\alpha u + \beta u^{-1}) + \delta u(u+1)(u-1)^{-1}$$

$$u_{xx} = (u^{-1} + (u-1)^{-1} + (u-x)^{-1})u_x^2/2 + \cdots$$

Bulk Scaling Limit of Gaussian Ensembles

$$E_\beta(0;s) = \mathbb{P}(\text{no levels lie in } (0,s))$$

(Gaudin '61, Mehta/des Cloizeaux '72)

$$E_2(0;s) = \det\left(\mathbb{1} - K\restriction_{L^2(0,s)}\right)$$

$$E_1(0;s) = \det\left(\mathbb{1} - K_+\restriction_{L^2(0,s/2)}\right)$$

$$E_4(0;s) = \cdots$$

with kernels

$$K(x,y) = \operatorname{sinc}(\pi(x-y))$$

$$K_+(x,y) = K(x,y) + K(x,-y)$$

(Jimbo/Miwa/Môri/Sato '80)

$$E_2(0;s) = \exp\left(\int_0^{\pi s} \frac{\sigma(x)}{x}\,dx\right)$$

$$E_1(0;s) = \exp\left(-\frac{1}{2}\int_0^{\pi s} \sqrt{-\frac{d}{dx}\frac{\sigma(x)}{x}}\,dx\right) E_2(0;s)^{1/2}$$

$$E_4(0;s) = \cdots$$

with σ-form of Painlevé V

$$(x\sigma_{xx})^2 = 4(\sigma - x\sigma_x)(x\sigma_x - \sigma + \sigma_x^2)$$

$$\sigma(x) \simeq -\frac{x}{\pi} - \frac{x^2}{\pi^2} \qquad (x \to 0)$$

Edge Scaling Limit of Gaussian Ensembles

$$F_\beta(s) = \mathbb{P}(\text{no levels lie in } (s,\infty))$$

(Mehta '91, Forrester '91, Ferrari/Spohn '05)

$$F_2(s) = \det\left(\mathbb{1} - K\restriction_{L^2(s,\infty)}\right)$$

$$F_1(s) = \det\left(\mathbb{1} - K_+\restriction_{L^2(s/2,\infty)}\right)$$

$$F_4(s) = \cdots$$

with kernels

$$K(x,y) = \frac{\operatorname{Ai}(x)\operatorname{Ai}'(y) - \operatorname{Ai}'(x)\operatorname{Ai}(y)}{x-y}$$

$$K_+(x,y) = \operatorname{Ai}(x+y)$$

(Tracy/Widom '94/'96)

$$F_2(s) = \exp\left(-\int_s^\infty (x-s)u(x)^2\,dx\right)$$

$$F_1(s) = \exp\left(-\frac{1}{2}\int_s^\infty u(x)\,dx\right) F_2(s)^{1/2}$$

$$F_4(s) = \cdots$$

with Hastings–McLeod solution of Painlevé II

$$u_{xx} = 2u^3 + xu$$

$$u(x) \simeq \operatorname{Ai}(x) \qquad (x \to \infty)$$

n-th Largest Level in Edge Scaled GUE

$$\mathbb{P}(\text{exactly } n \text{ levels lie in } (s,\infty)) = \frac{(-1)^n}{n!}\left(\frac{\partial}{\partial z}\right)^n F_2(s;z)\Big|_{z=1}$$

(Mehta '91, Forrester '91)

$$F_2(s;z) = \det\left(\mathbb{1} - z\,K\restriction_{L^2(s,\infty)}\right)$$

with kernel

$$K(x,y) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}$$

(Tracy/Widom '94)

$$F_2(s;z) = \exp\left(-\int_s^\infty (x-s)u(x;z)^2\,dx\right)$$

with Painlevé II

$$u_{xx} = 2u^3 + xu$$

$$u(x;z) \simeq \sqrt{z}\,\mathrm{Ai}(x) \qquad (x\to\infty)$$

... *much more involved for GOE and GSE*

$k \times k$-Laguerre Unitary Ensemble with Weight $x^n e^{-x}$

$$G_{k,n}(s) = \mathbb{P}(\text{no levels lie in } (0,s))$$

(Nagao/Wadati '91)

$$G_{k,n}(s) = \det\left(\mathbb{1} - K\restriction_{L^2(0,s)}\right)$$

with kernel

$$K(x,y) = \frac{\phi_{k-1}(x)\phi_k(y) - \phi_k(x)\phi_{k-1}(y)}{(k(k+n))^{-1/2}(x-y)}$$

$$\phi_k(x) = \sqrt{\frac{k!}{(k+n)!}}\, x^{n/2} e^{-x/2} L_k^{(n)}(x)$$

(Tracy/Widom '94)

$$G_{k,n}(s) = \exp\left(-\int_0^s \frac{\sigma(x)}{x}\,dx\right)$$

with Jimbo–Miwa–Okamoto σ-form of Painlevé V

$$(x\sigma_{xx})^2 = (\sigma - x\sigma_x - 2\sigma_x^2 + (2k+n)\sigma_x)^2 - 4\sigma_x^2(\sigma_x - k)(\sigma_x - k - n)$$

$$\sigma(x) \simeq \frac{k}{(n+1)!}\binom{k+n}{n} x^{n+1} \quad (x \to 0)$$

Gaudin's Method ('61) *(bulk scaling limit of GUE)*

$$E_2(0;2s) = \mathbb{P}(\text{no levels lie in } (0,2s))$$
$$= \det\left(\mathbb{1} - \frac{s}{2}K_s^\dagger K_s\restriction_{L^2(-1,1)}\right)$$
$$= \prod_{n=0}^{\infty}\left(1 - \frac{s}{2}|\lambda_n(K_s)|^2\right)$$

with kernel

$$K_s(x,y) = e^{i\pi sxy}$$

observe $[K_s, L_s] = 0$ for the differential operator

$$L_s u = -((1-x^2)u_x)_x + \pi^2 s^2 x^2 u$$
$$0 = (1-x^2)u(x)|_{x=\pm1} = (1-x^2)u'(x)|_{x=\pm1}$$

eigenfunctions of L_s are known as $u_n(x) = S_{n,0}^{(1)}(\pi s, x)$ *(radial prolate spheroidal wave functions)* $\rightsquigarrow$

$$\lambda_{2n}(K_s) = \frac{1}{u_{2n}(0)}\int_{-1}^{1} u_{2n}(\xi)\, d\xi$$
$$\lambda_{2n+1}(K_s) = \frac{i\pi s}{u'_{2n+1}(0)}\int_{-1}^{1} u_{2n+1}(\xi)\xi\, d\xi$$

*there is **no** such method for the edge scaling limit*

The Claim

Without the Painlevé representations, the numerical evaluation of the Fredholm determinants is quite involved. — Tracy/Widom '00

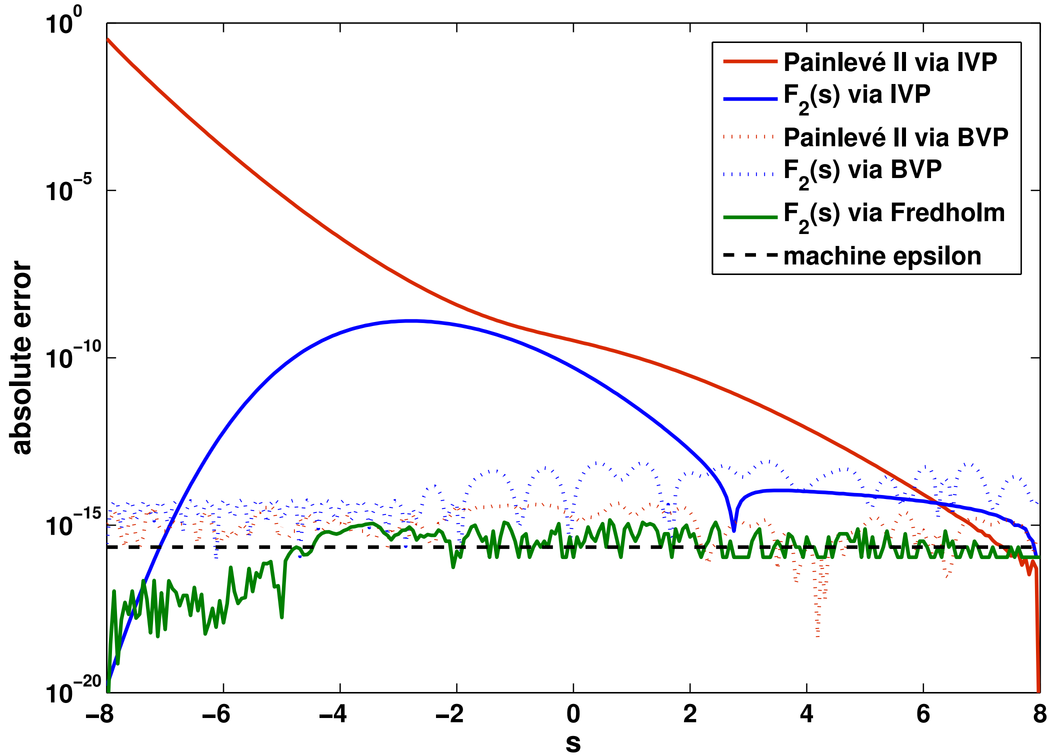
Challenging that Claim:

Evaluating Painlevé I–VI Is More Difficult than You Thought

$$u_{xx} = 2u^3 + xu, \qquad u(x) \simeq \theta \,\mathrm{Ai}(x) \quad (x \to \infty)$$

Numerical Evaluation of the Tracy–Widom Distribution $F_2(s)$

... there is yet no library software for the Painlevé transcendents



absolute error of various numerical approaches using IEEE double precision

- via Painlevé II as IVP *(backwards)*

  Prähofer ('04): 16 digits (1500 internally!)

  Bejan ('05): 3 digits

  Edelman/Persson ('05): 8 digits @ 8.9 sec

- via Painlevé II as BVP

  Tracy/Widom ('94): 12 digits (75 internally!)

  Dieng ('05): 9 digits

  Driscoll/B./Trefethen ('08): 13 digits @ 2.0 sec

- via Fredholm determinant

  B. ('08): 15 digits @ 1.4 sec

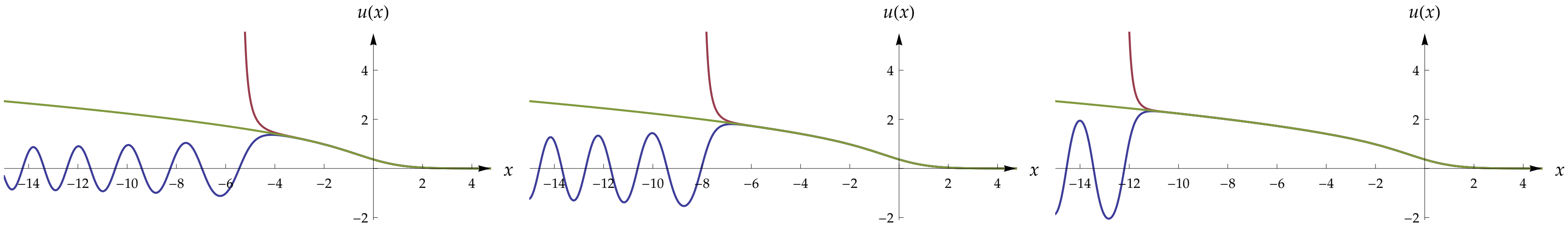
solution via Fredholm determinant: *much simpler, more efficient, and more accurate*

Explanation

solution of Painlevé II,

$$u_{xx} = 2u^3 + xu, \qquad u(x) \simeq \theta \,\mathrm{Ai}(x) \quad (x \to \infty),$$

is *separatrix* for $\theta = 1$ $\rightsquigarrow$ IVP highly unstable



$u(z)$ with $\theta = 1 - 10^{-4}, 1, 1 + 10^{-4}$ $\quad$ $u(z)$ with $\theta = 1 - 10^{-8}, 1, 1 + 10^{-8}$ $\quad$ $u(z)$ with $\theta = 1 - 10^{-16}, 1, 1 + 10^{-16}$

consequences

- F_2 via IVP solution of Painlevé II $\rightsquigarrow$ not more than 8 digits in IEEE arithmetic
- calculate F_2 via a BVP solution $\rightsquigarrow$ detailed *connection formulae* needed:

$$u(x) \simeq \theta \,\mathrm{Ai}(x) \quad (x \to \infty) \qquad \Rightarrow \qquad u(x) \simeq \,? \quad (x \to -\infty)$$

Painlevé II (Ablowitz/Segur '77, Hastings/McLeod '80, Its/Kapaev '89)

$$u_{xx} = 2u^3 + xu, \qquad u(x) \simeq \theta\,\mathrm{Ai}(x) \qquad (x \to \infty)$$

$\Rightarrow$ for $x \to -\infty$

- $0 < \theta < 1$ $\qquad u(x) \simeq c_\theta (-x)^{1/4} \cos\left(\frac{2}{3}(-x)^{3/2} + c'_\theta \log(8(-x)^{3/2}) + \phi_\theta\right)$
- $\theta = 1$ $\qquad u(x) \simeq \sqrt{-x/2}$
- $\theta > 1$ $\qquad u(x) \simeq (x - x_\theta)^{-1} \qquad (x \to x_\theta)$

σ-form of Painlevé V (McCoy/Tang '86, Basor/Tracy/Widom '92, Widom '94)

$$(x\sigma_{xx})^2 = 4(\sigma - x\sigma_x)(x\sigma_x - \sigma + \sigma_x^2), \qquad \sigma(x) \simeq -\theta\left(\frac{x}{\pi} + \frac{x^2}{\pi^2}\right) \qquad (x \to 0)$$

$\Rightarrow$ for $x \to \infty$

- $0 < \theta < 1$ $\qquad \sigma(x) \simeq \log(1-\theta)x/\pi$
- $\theta = 1$ $\qquad \sigma(x) \simeq -x^2/4$

Challenging that Claim:

Evaluating Fredholm's Det Is Much Easier than You Thought

$$\det\left(\delta_{ij} + z\, w_j K(x_i, x_j)\right)_{i,j=1}^{n}$$

```
[w,x] = QuadratureRule(a,b,n);
w = sqrt(w); [xi,xj] = ndgrid(x,x);
d = det(eye(n)+z*(w'*w).*K(xi,xj));
```

Determinants of Trace Class Operators K in a Hilbert space $\mathcal{H}$

(Grothendieck '56, Gohberg/Kreĭn '59, Dunford/Schwarz '63, Simon '77)

there are several equivalent definitions of the *entire* function $d(z) = \det(\mathbb{1} + zK)$

1. F_n finite rank, $F_n \to K$ in trace class norm

$$d(z) = \lim_{n\to\infty} \det(\mathbb{1} + zF_n)$$

2. $\lambda_1(K), \lambda_2(K), \lambda_3(K), \ldots$ eigenvalues of K (accounting multiplicities)

$$d(z) = \prod_{j=1}^{\infty}(1 + z\lambda_j(K))$$

3. analytic continuation

$$d(z) = \exp(\operatorname{tr}\log(\mathbb{1} + zK))$$

4. generalizing Fredholm's classical power series

$$d(z) = \sum_{m=0}^{\infty} z^m \operatorname{tr} \bigwedge^m K$$

Fredholm (1899)

Ivar Fredholm (1866–1927)

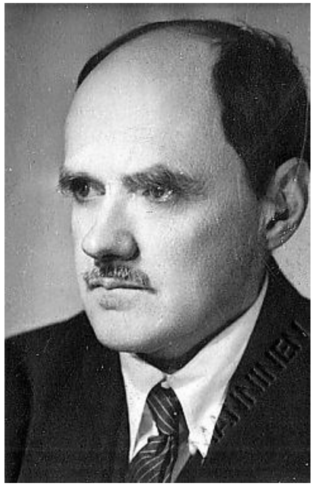
integral equation $(\mathbb{1} + zK)u = f$ of the 2nd kind, i.e.,

$$u(x) + z \int_a^b K(x,y)u(y)\,dy = f(x) \qquad (x \in (a,b))$$

uniquely solvable iff

$$\det(\mathbb{1} + zK) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \int_{[a,b]^m} \det\left(K(t_i,t_j)\right)_{i,j=1}^{m} dt \neq 0$$

Nyström (1930)



E. J. Nyström (1895–1960)

using an n-point quadrature formula Q (weights w_j, nodes x_j)
one gets $u_i \approx u(x_i)$ by solving the $n \times n$ linear system

$$u_i + z \sum_{j=1}^{n} w_j K(x_i,x_j) u_j = f(x_i) \qquad (i = 1,\ldots,n)$$

"New" Numerical Method (B. '08)

approximate $d(z) = \det(\mathbb{1} + zK)$ by the $n \times n$ determinant of Nyström's system:

$$\boxed{d_Q(z) = \det(\delta_{ij} + z\, w_j K(x_i, x_j))_{i,j=1}^n}$$

Theorem (B. '08)

K integral operator on $\mathcal{H} = L^2(a,b)$ with *continuous* kernel

Q quadrature formula of order ν with *positive* weights

- if kernel is $C^{k-1,1}([a,b]^2)$:

$$|d_Q(z) - d(z)| \leqslant c_{K,k}\, \nu^{-k} \qquad (\nu \to \infty)$$

- if kernel is *analytic* in a neighborhood of $[a,b]^2$, there is $\rho > 1$ s.t.:

$$|d_Q(z) - d(z)| \leqslant c_K\, \rho^{-\nu} \qquad (\nu \to \infty)$$

idea of proof: change the order of limits *(read the masters!)*

$$\det(\mathbb{1} + zK) \overset{\text{Fredholm}}{\underset{1903}{=}} \sum_{m=0}^{\infty} \frac{z^m}{m!} \int_a^b \cdots \int_a^b \det\left(K(t_i, t_j)\right)_{i,j=1}^m \, dt_1 \cdots dt_m$$

$$\overset{\text{Pólya}}{\underset{1933}{=}} \sum_{m=0}^{\infty} \lim_{n\to\infty} \frac{z^m}{m!} \sum_{k_1=1}^{n} \cdots \sum_{k_m=1}^{n} w_{k_1} \cdots w_{k_m} \cdot \det\left(K(x_{k_i}, x_{k_j})\right)_{i,j=1}^m$$

$$\overset{\text{Hilbert}}{\underset{1904}{=}} \lim_{n\to\infty} \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{k_1=1}^{n} \cdots \sum_{k_m=1}^{n} w_{k_1} \cdots w_{k_m} \cdot \det\left(K(x_{k_i}, x_{k_j})\right)_{i,j=1}^m$$

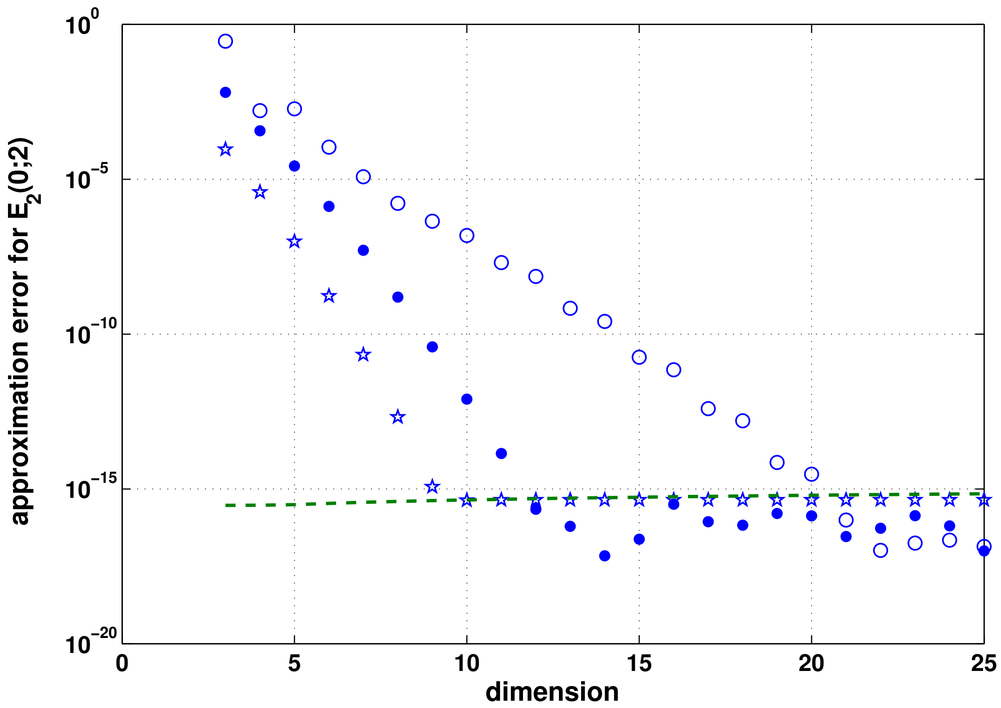
$$= \lim_{n\to\infty} \sum_{m=0}^{n} \frac{z^m}{m!} \sum_{k_1=1}^{n} \cdots \sum_{k_m=1}^{n} \det\left((K_n)_{k_i,k_j}\right)_{i,j=1}^m \overset{\text{v. Koch}}{\underset{1892}{=}} \lim_{n\to\infty} \det(\mathbb{1}_n + zK_n)$$

with the $n \times n$-matrix

$$K_n = \left(w_j K(x_i, x_j)\right)_{i,j=1}^n$$

Bulk Scaling Limit of GUE

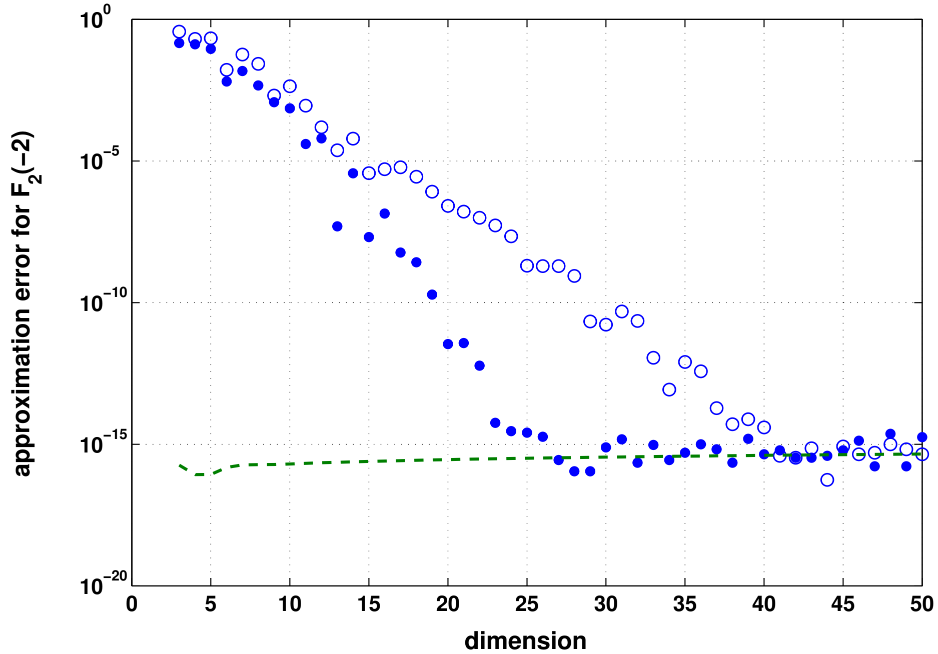
$$E_2(0;s) = \det\left(\mathbb{1} - K\restriction_{L^2(0,s)}\right), \qquad K(x,y) = \operatorname{sinc}(\pi(x-y))$$



stars: Gaudin's method, dots: Gauss–Legendre quadrature, circles: Clenshaw–Curtis quadrature

Edge Scaling Limit of GUE *(Tracy–Widom distribution)*

$$F_2(s) = \det\left(\mathbb{1} - K\restriction_{L^2(s,\infty)}\right), \qquad K(x,y) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}$$



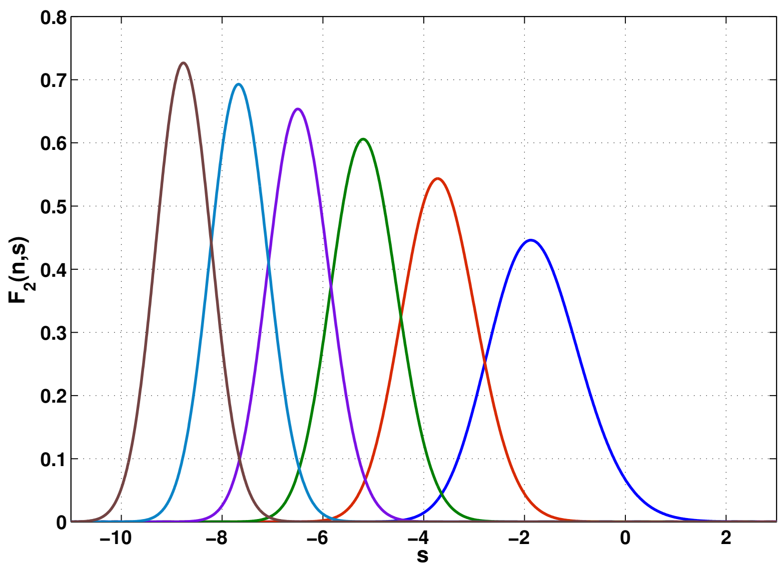
dots: Gauss–Legendre quadrature, circles: Clenshaw–Curtis quadrature

Perturbation bound for n-dimensional determinants: (B. '08)

$$\text{round-off error} \leqslant \sqrt{n}\,\|K_n\|_F \cdot u_{\text{machine}}$$

n-th Largest Level in Edge Scaled GUE

$$\mathbb{P}(\text{exactly } n \text{ levels lie in } (s,\infty)) = \frac{(-1)^n}{n!}\left(\frac{\partial}{\partial z}\right)^n \det\left(\mathbb{1} - zK\restriction_{L^2(s,\infty)}\right)\Big|_{z=1}$$



probability density of n-th largest level in edge scaled GUE

Numerical Method

$f(z) = \det(\mathbb{1} + zK)$ is *entire*

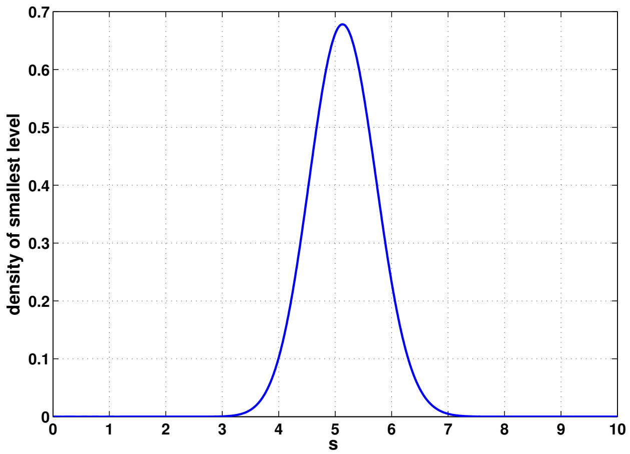
$\rightsquigarrow$ Cauchy's formula applies

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi r^n}\int_0^{2\pi} e^{-in\theta} f(z + re^{i\theta})\, d\theta$$

- Trapezoidal rule *exponentially* convergent
- numerical stability: judicious choice of $r > 0$

$k \times k$-LUE with Weight $x^n e^{-x}$ ($k = 80$, $n = 40$, to be concrete)

$$\mathbb{P}(\text{no levels lie in } (0,s)) = \det\left(\mathbb{1} - K\restriction_{L^2(0,s)}\right)$$



probability density of minimal level in 80×80-LUE with $n = 40$

$$K(x,y) = \frac{\phi_{k-1}(x)\phi_k(y) - \phi_k(x)\phi_{k-1}(y)}{(k(k+n))^{-1/2}(x-y)}$$

$$\phi_k(x) = \sqrt{\frac{k!}{(k+n)!}}\, x^{n/2} e^{-x/2} L_k^{(n)}(x)$$

$\mu = 5.14156\,81318\cdots$ [5.141 *from* 10^4 *samples*]

$\sigma^2 = 0.34347\,52478\cdots$ [0.339 — '' —]

compare with Painlevé V approach: $(x\sigma_{xx})^2 = (\sigma - x\sigma_x - 2\sigma_x^2 + (2k+n)\sigma_x)^2 - 4\sigma_x^2(\sigma_x - k)(\sigma_x - k - n)$

$$\mathbb{P}(s) = \exp\left(-\int_0^s \frac{\sigma(x)}{x}\,dx\right) \qquad \sigma(x) \simeq \frac{k}{(n+1)!}\binom{k+n}{n} x^{n+1} \quad (x \to 0)$$

connection formula (Forrester/Witte '02): $\sigma(x) \simeq k(x-n) + k^2 n x^{-1} \quad (x \to \infty)$

Matrix Kernels ...

$$\det\left(\mathbb{1} + z\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\restriction_{L^2(s_1,\infty)\oplus L^2(s_2,\infty)}\right)$$

Edge Scaling Limit of GUE Matrix Diffusion = Airy$_2$ Process

$M_n(t)$ $n \times n$-Hermitean-matrix valued process, coefficients Ornstein–Uhlenbeck

$$\mathcal{A}_2(t) = \lim_{n\to\infty} \frac{\lambda_{\max}\left(M_n(n^{-1/3}t)\right) - \sqrt{2n}}{2^{-1/2}n^{-1/6}}$$

relation to PNG (polynuclear growth) droplet model

$\rightsquigarrow$ joint probablity distribution (Prähofer/Spohn ’02)

$$\mathbb{P}(\mathcal{A}_2(t) \leqslant s_1, \mathcal{A}_2(0) \leqslant s_2) = \det\left(\mathbb{1} - \begin{pmatrix} K_0 & K_t \\ K_{-t} & K_0 \end{pmatrix}\restriction_{L^2(s_1,\infty)\oplus L^2(s_2,\infty)}\right)$$

with kernel

$$K_t(x,y) = \begin{cases} \displaystyle\int_0^\infty e^{-\xi t}\,\mathrm{Ai}(x+\xi)\,\mathrm{Ai}(y+\xi)\,d\xi & t > 0 \\ \displaystyle-\int_{-\infty}^0 e^{-\xi t}\,\mathrm{Ai}(x+\xi)\,\mathrm{Ai}(y+\xi)\,d\xi & t \leqslant 0 \end{cases}$$

Adler/van Moerbeke ('05)

$G(t,x,y) = \log \mathbb{P}(\mathcal{A}_2(t) \leqslant x, \mathcal{A}_2(0) \leqslant y)$ satiesfies nonlinear 3rd order PDE

$$
\begin{aligned}
t\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) G &= \frac{\partial^3 G}{\partial x^2 \partial y}\left(2\frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial x \partial y} - \frac{\partial^2 G}{\partial x^2} + x - y - t^2\right) \\
&\quad - \frac{\partial^3 G}{\partial y^2 \partial x}\left(2\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial x \partial y} - \frac{\partial^2 G}{\partial y^2} - x + y - t^2\right) \\
&\quad + \left(\frac{\partial^3 G}{\partial x^3}\frac{\partial}{\partial y} - \frac{\partial^3 G}{\partial y^3}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) G
\end{aligned}
$$

$\rightsquigarrow$ asymptotic expansions, e.g.:

$$
\mathbb{P}(\mathcal{A}_2(t) \leqslant x, \mathcal{A}_2(0) \leqslant y) = F_2(x)F_2(y) + \frac{F_2'(x)F_2'(y)}{t^2} + O(t^{-4})
$$

aside: useful for numerical calculations? most probably not ...

Edge Scaling Limit of GOE Matrix Diffusion = Airy$_1$ Process ?

$M_n(t)$ $n \times n$-symmetric-matrix valued process, coefficients Ornstein–Uhlenbeck

$$\mathcal{A}_1(t) = \lim_{n\to\infty} \frac{\lambda_{\max}\left(M_n(2n^{-1/3}t)\right) - \sqrt{n}}{n^{-1/6}}$$

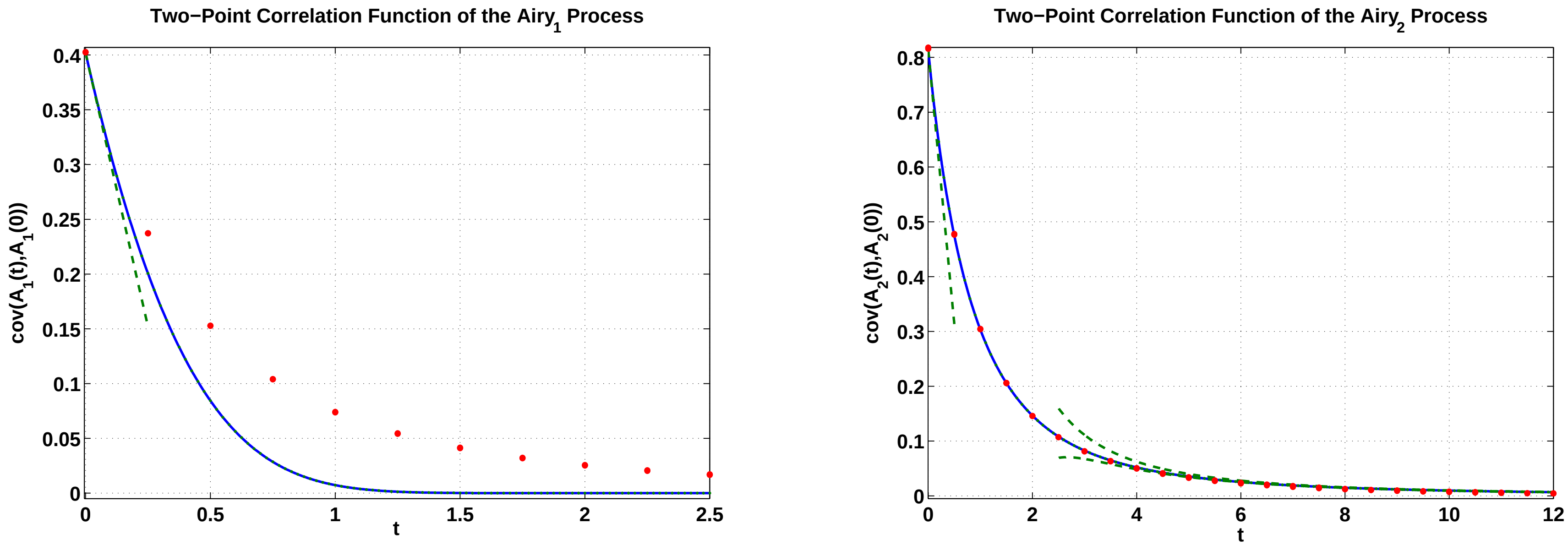
(Sasamoto '05, Borodin/Ferrari/Prähofer/Sasamoto '07)

conjectured relation to the flat PNG model (universality!) $\rightsquigarrow$

$$\mathbb{P}(\mathcal{A}_1(t) \leqslant s_1, \mathcal{A}_1(0) \leqslant s_2) \overset{?}{=} \det\left(\mathbb{1} - \begin{pmatrix} K_0 & K_t \\ K_{-t} & K_0 \end{pmatrix}\restriction_{L^2(s_1,\infty)\oplus L^2(s_2,\infty)}\right)$$

with kernel

$$K_t(x,y) = \begin{cases} \operatorname{Ai}(x+y+t^2)e^{t(x+y)+2t^3/3} - \dfrac{\exp(-(x-y)^2/(4t))}{\sqrt{4\pi t}} & t > 0 \\ \operatorname{Ai}(x+y+t^2)e^{t(x+y)+2t^3/3} & t \leqslant 0 \end{cases}$$



red: Monte–Carlo for matrix size $N = 128$ and $N = 256$; green: known asymptotics; blue: numerical calculation with absolute precision $5 \cdot 10^{-11}$

$$\operatorname{cov}(\mathcal{A}_k(t), \mathcal{A}_k(0)) = \mathbb{E}(\mathcal{A}_k(t)\mathcal{A}_k(0)) - \mathbb{E}(\mathcal{A}_k(t))\mathbb{E}(\mathcal{A}_k(0))$$

$$= \int_{\mathbb{R}^2} s_1 s_2 \frac{\partial^2 \mathbb{P}(\mathcal{A}_k(t) \leqslant s_1, \mathcal{A}_k(0) \leqslant s_2)}{\partial s_1 \partial s_2}\, ds_1 ds_2 - \mathbb{E}(F_k)^2$$

Conclusion: (B./Ferrari/Prähofer ’08)

limit of GOE matrix diffusion $\neq$ Airy$_1$ process *(most probably, so)*

n-th Largest Level in Edge Scaled GSE

$$\mathbb{P}(\text{exactly } n \text{ levels lie in } (s,\infty)) = E_4(n;s) = \frac{(-1)^n}{n!}\left(\frac{\partial}{\partial z}\right)^n F_4(s;z)\Big|_{z=1}$$

(Forrester/Nagao/Honner '99, Tracy/Widom '04)

$$F_4(s/\sqrt{2};z) = \sqrt{\det\left(\mathbb{1} - \frac{z}{2}\begin{pmatrix} S(x,y) & SD(x,y) \\ IS(x,y) & S(y,x)\end{pmatrix}\restriction_{L^2(s,\infty)\oplus L^2(s,\infty)}\right)}$$

$$S(x,y) = K_2(x,y) - \frac{1}{2}\operatorname{Ai}(x)\int_y^\infty \operatorname{Ai}(\eta)\,d\eta$$

$$SD(x,y) = -\partial_y K_2(x,y) - \frac{1}{2}\operatorname{Ai}(x)\operatorname{Ai}(y)$$

$$IS(x,y) = -\int_x^\infty K_2(\xi,y)\,d\xi + \frac{1}{2}\int_x^\infty \operatorname{Ai}(\xi)\,d\xi \int_y^\infty \operatorname{Ai}(\eta)\,d\eta$$

$$K_2(x,y) = \frac{\operatorname{Ai}(x)\operatorname{Ai}'(y) - \operatorname{Ai}'(x)\operatorname{Ai}(y)}{x-y}$$

A New Formula from Numerical Experiments (B. '09)

$$F_{\pm}(s;z) = \det\left(\mathbb{1} \mp \sqrt{z}\, K\restriction_{L^2(s/2,\infty)}\right), \qquad K(x,y) = \mathrm{Ai}(x+y)$$

yields *(Ferrari/Spohn '05)*

$$F_4(s/\sqrt{2};1) = \frac{1}{2}\left(F_+(s;1) + F_-(s;1)\right)$$
$$F_2(s;z) = F_+(s;z) \cdot F_-(s;z)$$

How about

$$\boxed{F_4(s/\sqrt{2};z) = \frac{1}{2}\left(F_+(s;z) + F_-(s;z)\right),}$$ then?

there is no obvious reason for it ...,

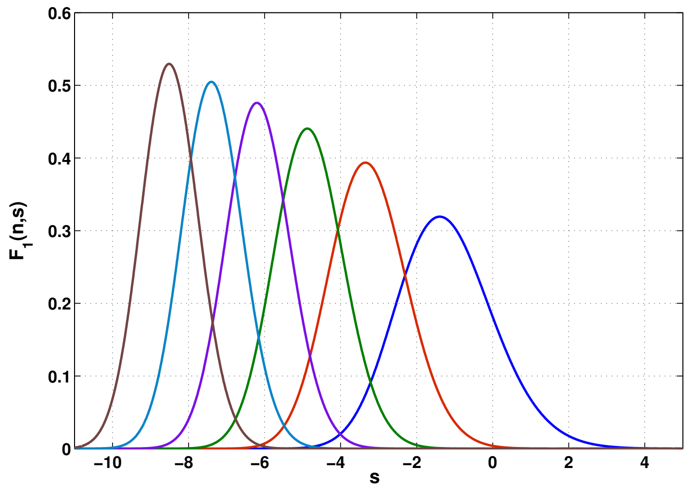
... but numerical tests with random s and z indicate the formula to be *true*

later, proof via Painlevé II representation (B. '09, Forrester '06)

n-th Largest Level in Edge Scaled GOE

$$E_\pm(n;s) = \frac{(-1)^n}{n!}\left(\frac{\partial}{\partial z}\right)^n \det\left(\mathbb{1} \mp \sqrt{z}\,K\restriction_{L^2(s/2,\infty)}\right)\Big|_{z=1}, \qquad K(x,y) = \mathrm{Ai}(x+y)$$

yields



probability density of n-th largest level in edge scaled GOE

$$E_1(2n;s) = E_+(n;s) + \sum_{k=1}^{n} \frac{\binom{2k-1}{k}}{4^k}\left(E_+(n-k,s) - E_-(n-k,s)\right)$$

$$E_1(2n+1;s) = \frac{E_+(n,s) + E_-(n,s)}{2} - E_1(2n,s)$$

technique: elimination process using interrelations (Forrester/Rains '01) between GOE, GUE, and GSE

$$\mathrm{GUE}_n = \mathrm{even}(\mathrm{GOE}_n \cup \mathrm{GOE}_{n+1}), \qquad \mathrm{GSE}_n = \mathrm{even}(\mathrm{GOE}_{2n+1})$$

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