ON BELLMAN'S LINEAR SEARCH PROBLEM

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Let $\mathcal{S}$ be the set of turning point sequences $x = (x_k)_{k\in\mathbb{Z}}$ satisfying

$$\cdots < -x_{2k+1} < -x_{2k-1} < \cdots < 0 < \cdots < x_{2k} < x_{2k+2} < \cdots .$$

The linear search problem (see Alpern and Gal 2003, §8.2.1) amounts for solving the minimax problem

$$V = \inf_{x\in\mathcal{S}} \sup_{n\in\mathbb{Z}} \frac{\sum_{k=-\infty}^{n+1} x_k}{x_n}. \tag{1}$$

Gal (1980, Chap. 6) developed a general theory for dealing with that kind of minimax problems (using advanced concepts such as, e.g., submartingales). However, for the specific problem at hand, Beck and Newman (1970, pp. 421–422) approached the minimax problem by using elementary methods only. In this note, we present an even simpler version of their proof.

Theorem. *The value of the minimax problem (1) is $V = 4$, which is attained if and only if $x_k = \alpha 2^k$, where α is a positive constant.*

Proof. Upon calculating

$$\frac{\sum_{k=-\infty}^{n+1} 2^k}{2^n} = 4 \qquad (n \in \mathbb{Z}),$$

we get $V \leqslant 4$. To show $V \geqslant 4$, let us assume to the contrary that, for some $x \in \mathcal{S}$,

$$\sup_{n\in\mathbb{Z}} \frac{\sum_{k=-\infty}^{n+1} x_k}{x_n} \leqslant \gamma, \qquad \text{that is,} \quad \sum_{k=-\infty}^{n+1} x_k \leqslant \gamma x_n \quad (n \in \mathbb{Z}), \tag{2}$$

with $\gamma = 4 - \beta$, $\beta > 0$. Set

$$y_n = \frac{1}{2^n} \sum_{k=-\infty}^{n+1} x_k.$$

Then $y_n > 0$ and

$$\begin{aligned} 2^{n+2} \cdot \frac{1}{2}(y_{n-1} + y_{n+1}) &= 4 \sum_{k=-\infty}^{n} x_k + \sum_{k=-\infty}^{n+2} x_k \leqslant 4 \sum_{k=-\infty}^{n} x_k + \gamma x_{n+1} \\ &= 4 \sum_{k=-\infty}^{n+1} x_k - \beta x_{n+1} \leqslant 4 \sum_{k=-\infty}^{n+1} x_k - \gamma^{-1}\beta \sum_{k=-\infty}^{n+2} x_k = 2^{n+2} y_n - 2^{n+1}\gamma^{-1}\beta y_{n+1}, \end{aligned}$$

so that

$$\frac{1}{2}(y_{n-1} + y_{n+1}) \leqslant y_n - \frac{1}{2}\gamma^{-1}\beta y_{n+1} \leqslant y_n. \tag{3}$$

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Thus, the sequence $(y_n)_{n\in\mathbb{Z}}$ is positive and concave and therefore, by the Lemma below, necessarily *constant*: $y_n = \eta$. By plugging this constant into (3), we arrive at

$$0 < \eta \leqslant \eta - \frac{1}{2}\gamma^{-1}\beta\eta,$$

a contradiction.

Finally, if the value V is attained for some $x \in \mathcal{S}$, then (2) holds with $\beta = 0$. As the argument that led us to (3) applies here, too, we infer that $y_n = 4\alpha$ for some constant $\alpha > 0$. Thus,

$$\sum_{k=-\infty}^{n} x_k = \alpha 2^{n+1} \qquad (n \in \mathbb{Z}),$$

and, by taking differences,

$$x_n = \alpha(2^{n+1} - 2^n) = \alpha 2^n \qquad (n \in \mathbb{Z}),$$

which concludes the proof. □

Lemma. *If the positive sequence $(y_n)_{n\in\mathbb{Z}}$ is concave, that is, if $y_n > 0$ satisfies*

$$\frac{y_{n-1} + y_{n+1}}{2} \leqslant y_n \qquad (n \in \mathbb{Z}), \tag{4}$$

then $y_n = const.$

Proof. The concavity condition (4) is equivalent to the monotonicity

$$y_{n+1} - y_n \leqslant y_n - y_{n-1} \qquad (n \in \mathbb{Z}).$$

By induction, we get

$$y_{n+k} \leqslant y_n + k(y_{n+1} - y_n) \qquad (n \in \mathbb{Z}, k \geqslant 0),$$

as there is

$$\begin{aligned} y_{n+k+1} &= (y_{n+k+1} - y_{n+k}) + y_{n+k} \\ &\leqslant (y_{n+1} - y_n) + y_n + k(y_{n+1} - y_n) = y_n + (k+1)(y_{n+1} - y_n). \end{aligned}$$

If we had $y_{n+1} < y_n$ for some n, we would infer that

$$y_{n+k} \leqslant y_n + k(y_{n+1} - y_n) \to -\infty \qquad (k \to \infty),$$

contradicting the positivity assumption. Thus, $y_{n+1} \geqslant y_n$ for all $n \in \mathbb{Z}$: the sequence (y_n) is increasing. Likewise, since $(y_{-n})_{n\in\mathbb{Z}}$ is positive and concave, too, we obtain that (y_{-n}) is increasing or, equivalently, that (y_n) is decreasing: a sequence, however, that is increasing and decreasing at the same time must be constant. □

References

Alpern, S. and Gal, S.: 2003, *The theory of Search Games and Rendezvous*, Kluwer Academic Publishers, Boston.

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