ON HATHAWAY’S CIRCULAR PURSUIT PROBLEM

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In the January 1920 issue of the American Mathematical Monthly, Arthur Hathaway (1855–1934) posed the following problem:

A dog at the center of a circular pond makes straight for a duck which is swimming along the edge of the pond. If the rate of swimming of the dog is to the rate of swimming of the duck as \( k : 1 \), determine the equation of the curve of pursuit and the distance the dog swims to catch the duck.

For the history of that problem see Hathaway (1921) or Nahin (2007, §2.1). There is no closed analytic expression for the solution, neither for the curve of pursuit nor for the distance to swim. Hence, given the initial positions of duck and dog, one has to rely on numerical methods to visualize the curve of pursuit (see Figs. 2–4). Whether, however, or not the dog eventually catches the duck can be (and should be) answered by mathematical proof; in fact, Hathaway (1921) showed:

- the dog catches the duck in finite time if and only if \( k > 1 \);
- if \( k \leq 1 \), the ratio of the lag distance to the radius of the pond approaches, as time goes to infinity, the value \( \sqrt{1 - k^2} \);
- specifically, though there will be no capture for \( k = 1 \), the dog is getting as near to the duck as one pleases.

In this note, upon focusing on the question of capturability, we shall show that Hathaway’s (1921) original arguments can be considerably simplified.\(^1\)

1. Notation

By choosing appropriate units of length and time, we arrange for the pond to have radius 1 and for the duck to have velocity 1; the velocity of the dog is then given by \( k \). The position of the duck (which is moving counter-clockwise) is denoted by \( P \); that of the dog by \( Q \); the center of the pond is \( C \). We express \( Q \)’s motion relative to \( P \), that is, the apparent motion of \( Q \) as seen by \( P \), in polar coordinates \((r, \phi)\) with respect to the axis \( PC \) (see Fig. 1),

\[
r = |PQ|, \quad \phi = \angle CPQ.
\]

Observe that, for \( Q \) within the pond,

\[
-\frac{\pi}{2} < \phi < \frac{\pi}{2} \quad (Q \neq P). \tag{1.1}
\]

\(^1\)Nahin (2007) does not really attempt such a proof and presents a derivation of the kinematic equations that is unnecessarily involved.

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Derivatives with respect to time $t$ will be denoted by a dot, namely

$$\dot{r} = \frac{dr}{dt}, \quad \dot{\phi} = \frac{d\phi}{dt}, \quad \text{etc.}$$

2. Kinematic Equations

The dog making it straight for the duck means that, in a fixed frame of reference, $Q$’s velocity vector always points directly to $P$. Therefore, the velocity with which $Q$ gains on $P$ (that is, $-\dot{r}$) is equal to his own velocity $k$ diminished by the projection of $P$’s velocity to $QP$, namely

$$\dot{r} = \sin \phi - k.$$  

(2.1)

By introducing $\rho = |QC|$ and projecting $Q$’s velocity to $QC$, we obtain

$$\dot{\rho} = k \sin \angle KCQ = k \frac{|QK|}{\rho} = k \frac{\cos \phi - r}{\rho}.$$  

(2.2)

Observe that the sign is correct if $\cos \phi - r < 0$, too. Alternatively, by the law of cosines, we get

$$\rho^2 = 1 + r^2 - 2r \cos \phi,$$

which, after differentiation with respect to $t$, yields

$$\rho \dot{\rho} = (r - \cos \phi) \dot{r} + r \dot{\phi} \sin \phi.$$

Hence, by equating the two different expressions for $\rho \dot{\rho}$ and using (2.1),

$$k(\cos \phi - r) = (r - \cos \phi) \dot{r} + r \dot{\phi} \sin \phi = (r - \cos \phi)(\sin \phi - k) + r \dot{\phi} \sin \phi,$$

which finally gives\(^2\)

$$r \dot{\phi} = \cos \phi - r.$$  

(2.3)

\(^2\)Compare our derivation of (2.1) and (2.3) with the six pages’ argument of Nahin (2007, pp. 44–51) that leads to the equivalent equations of motion (2.1.8) and (2.1.10); see also his (2.2.11) and (2.2.12). Note that Nahin’s $\rho$ is our $r$, his $n$ is our $k$, and that for us $a = 1$. 

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Fig. 1. The basic geometry for the kinematic equations.
3. The Case $k > 1$: Dog Catches Duck

If $k > 1$, the kinematic equation (2.1) permits of the simple bound

$$\dot{r} = \sin \phi - k \leq 1 - k < 0.$$  

Therefore, as the distance $r$ decreases steadily with at least the rate $1 - k$, capture will be accomplished at a finite time $t_* < \infty$, that is, $r(t_*) = 0$. Integration yields

$$-r(0) = r(t_*) - r(0) = \int_0^{t_*} \dot{r}(t) \, dt \leq \int_0^{t_*} (1 - k) \, dt = (1 - k)t_*;$$

hence, the time of capture is bounded by

$$0 < t_* \leq \frac{r(0)}{k - 1}.$$  

Note that the bound $r(0)/(k - 1)$ would give us the exact time of capture for a pursuit and evasion game taking place on a straight line.

Remark. We shall show in Step 4 of §4 that the duck is always caught from behind: $\phi(t_*) = \pi/2$.

4. The Case $k = 1$: No Capture, but an Arbitrarily Close Approach

In general, there are two mutually exclusive possibilities: either $r(t_*) = 0$ for some finite time $t_*$ or $r(t) > 0$ for all $t > 0$, in which case we simply put $t_* = \infty$.

If $k = 1$, the distance $r$ decreases monotonically, as by (2.1)

$$\dot{r} = \sin \phi - 1 \leq 0.$$  

Therefore, the limit

$$\lim_{t \to t_*} r(t) = r_* \geq 0$$

exists (by definition, $r_* = 0$ if $t_* < \infty$). We shall show, by breaking the proof into several steps, that $t_* = \infty$ but nevertheless $r_* = 0$. 

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**Fig. 2.** The case $k > 1$: Q catches P in finite time (always from behind).
To prove, in passing, the assertion of the last remark of §3, we begin with assuming just \( k \geq 1 \); from Step 5 onwards, we shall get to the specifics of \( k = 1 \).

**Step 1.** The kinematic equation (2.3) teaches us that \( \phi \) is strictly increasing, or strictly decreasing, depending on whether the sign of \( u = \cos \phi - r \) is positive or negative. We observe that \( u = 0 \) if \( Q \) is on the edge of the disk \( D \) with diameter \( PC \) (see Fig. 1), and that \( u > 0 \) \((u < 0)\) if \( Q \) is in the interior \((\text{exterior})\) of that disk. By using the kinematic equations (2.1) and (2.3), we calculate the derivative

\[
\dot{u} = k - \frac{\cos \phi \sin \phi}{r}.
\]

If \( Q \neq P \) is on the edge of the disk \( D \) \(\text{(i.e.,} u = 0)\), this simplifies to

\[
\dot{u} = k - \sin \phi > 0,
\]

as \( k \geq 1 \) and (1.4). That is, \( Q \) must move from the edge into the interior of \( D \) \(\text{(to positive values of} u)\). In particular, once having entered it, \( Q \) will never leave \( D \).

**Step 2.** By comparing (2.2) with (2.3), we realize that

\[
\rho \dot{\rho} = kr \phi;
\]

hence, according to Step 1, \( \rho \) and \( \phi \) share their monotonicity properties: both quantities are strictly increasing \((\text{decreasing})\) in the interior \((\text{exterior})\) of \( D \).

**Step 3.** \( Q \) will eventually be found in the interior of \( D \). For, if otherwise, \( \phi \) and \( \rho \) would be, by Step 2, strictly monotonically decreasing throughout the pursuit \(\text{(in particular, we could assume that} Q \text{had started at some} -\pi/2 < \phi_0 < \pi/2 \text{and} 0 \leq \rho_0 < 1)\). Hence, for \( k \geq 1 \),

\[
\dot{r} = \sin \phi - k \leq \sin \phi_0 - k < 0,
\]

which would lead, as in §3, to a capture in a finite time \( t_* < \infty \). But, \( r \to 0 \) as \( t \to t_* \) would imply \( \rho \to 1 \); therefore, we would eventually reach some \( \rho > \rho_0 \), which is impossible, though, unless \( \rho \) had *increased* at some point.
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Step 4. Let us assume that \( t_* < \infty \) (which is, as we know, the case for \( k > 1 \); for \( k = 1 \) it will lead to a contradiction in Step 5). By definition, \( t_* < \infty \) describes capture in finite time: \( r_* = 0 \), that is, \( Q \to P \) as \( t \to t_* \). Therefore, as the chord \( QP \) of the apparent path of \( Q \) ending in \( P \) is vanishing, its angle \( \psi \) to the tangent of that path at \( Q \) must also be vanishing: \( \psi \to 0 \). By expressing \( \psi \) in terms of the polar coordinates, and by the kinematic equations (2.1) and (2.3), we obtain

\[
\tan \psi = \frac{r \dot{\phi}}{\phi} = \frac{\cos \phi - r}{\sin \phi - k}.
\]

Hence, \( r \to 0 \) and \( \psi \to 0 \) imply that \( \cos \phi \to 0 \). Since \( \phi \) is eventually monotonically increasing (see Step 3), we thus get \( \phi \to \pi/2 \): the capture is always from behind.

Step 5. If \( k = 1 \), the assumption \( t_* < \infty \) leads to a contradiction. Given a time \( t < t_* \) close enough, the mean value theorem and the eventual monotonicity of \( \phi \) yield, for some \( t < t' < t_* \),

\[
\frac{r_* - r(t)}{t_* - t} = \dot{r}(t') = \sin \phi(t') - k \geq \sin \phi(t) - k.
\]

Thus, by (4.1) and because of \( k = 1 \),

\[
t_* - t \geq \frac{r}{k - \sin \phi} = \frac{r - \cos \phi}{k - \sin \phi} + \frac{\cos \phi}{k - \sin \phi} = \tan \psi + \frac{\cos \phi}{1 - \sin \phi}.
\]

As \( t \to t_* \), \( \phi \to 0 \), and \( \phi \to \pi/2 \), the last term may be made as large as we please, which finally leads to the advertised contradiction.

Step 6. Knowing that \( t_* = \infty \) if \( k = 1 \), we shall now conclude that \( r_* = 0 \) then, too. Since \( \phi \) is eventually monotonically increasing (see Step 3), the limit \( \phi \to \phi_* \) exists and, furthermore, \( \phi \leq \phi_* \) for \( t \) large enough. If \( \phi_* < \pi/2 \), the kinematic equation (2.1) would eventually yield

\[
\dot{r} \leq \sin \phi_* - k < 0,
\]

which would allow us, as so often before, to infer a capture in finite time \( t_* < \infty \). Hence \( \phi_* = \pi/2 \), which implies, however, the assertion \( r_* = 0 \): for there is no point in the pond with \( \phi = \pi/2 \) other than \( P \).

5. The Case \( k < 1 \): Settling down to a Limit Cycle

Here, the equations

\[
\sin \phi_* = k, \quad r_* = \cos \phi_* = \sqrt{1 - k^2}, \quad (5.1)
\]

define the unique coordinates \((r_*, \phi_*)\) of a distinguished point \( Q_* \neq P \), located on the edge of the disk \( D \): the kinematic equations show that a dog, starting from \( Q_* \), would make — as the duck \( P \) sees it — no apparent motion at all (such points are called stationary). Then, in the absolute frame of reference, the dog moves at the edge of a disk with radius \( k \), lagging a distance \( \sqrt{1 - k^2} \) behind the duck. We shall show that this is, in general, the limit behavior as \( t \to \infty \) (see Fig. 4, which also demonstrates that the dog may actually come fairly close to the duck before it is settling down to its circular limit cycle).

As in §4, we break the proof into several steps. The first step is, in fact, just an easy modification of Steps 1–3 of §4.
Step 1. By (2.1), \( r \) is strictly increasing (decreasing) if \( \phi > \phi_* \) (\( \phi < \phi_* \)); by (2.3), \( \phi \) is strictly increasing (decreasing) in the interior (exterior) of \( D \). Arguing as in Step 1 of §4, we see that \( D \) can only be entered at \( \phi < \phi_* \), and can only be left at \( \phi > \phi_* \). Accordingly, arguing as in Step 3 of §4, we see that \( Q \), if at \( \phi < \phi_* \), will eventually be drawn into \( D \) (while possibly, however, leaving it later on again).

Step 2. There will be no capture \( r \to 0 \) in finite time \( t \to t_* \). Otherwise, as in Step 4 of §4, we would have \( \cos \phi \to 0 \); leaving us with but two possibilities: capture from behind (\( \phi \to \pi/2 \)) and capture from ahead (\( \phi \to -\pi/2 \)).

For a capture from behind, the positive quantity \( r \) would, by Step 1, eventually be strictly increasing, contradicting \( r \to 0 \).

For a capture from ahead, restricting ourselves to sufficiently large \( t \), we may assume \( \phi < \phi_* \) for all times to come. By Step 1, \( Q \) would eventually be drawn into the disk \( D \); yet, because of \( \phi < \phi_* \), it could not leave that disk any more (where \( \phi \) is strictly increasing), contradicting \( \phi \to -\pi/2 \).

Step 3. Looking at the squared euclidian distance of \( Q \) to \( Q_* \),
\[
L = |Q-Q_*|^2 = (r \cos \phi - r_* \cos \phi_*)^2 + (r \sin \phi - r_* \sin \phi_*)^2 \geq 0,
\]
we calculate its derivative, by using the kinematic equations together with (5.1):
\[
\dot{L} = -4k(r + r_*) \sin((\phi - \phi_*)/2)^2 \leq 0. \tag{5.2}
\]
Hence, \( L \) is monotonically decreasing and the limit \( L \to L_* \geq 0 \) exists as \( t \to \infty \). The next step will rule out the possibility that \( L_* > 0 \); therefore \( L_* = 0 \), that is, \( r \to r_* \) and \( \phi \to \phi_* \) as asserted.

Step 4 (advanced). We shall now show that the assumption \( L_* > 0 \) leads to a contradiction. To do so, we start using more advanced tools. By the last step, we already know that the apparent motion of \( Q \) gets, as \( t \to \infty \), arbitrarily close to the circle \( \Lambda \) of radius \( L_*^{1/2} \) centered at \( Q_* \). Now, the Poincaré–Bendixson theory

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3People educated in dynamical systems generally call functions \( L \) that satisfies \( \dot{L} \leq 0 \) along the path of motion a Lyapunov function. It is used to prove the stability of stationary points such as \( Q_* \).
(Hartman 1982, Chap. VII) teaches us that either $Q$ approaches a stationary point, or that $\Lambda$ itself is a legitimate apparent path of motion (that is, if $Q$ starts on $\Lambda$, it will keep cycling on that circle). Since $Q_*$ is the unique stationary point of the kinematic equations, the first possibility does not apply and we are left with the second. Cycling, however, on $\Lambda$ is impossible, too, as there are, by (5.2), just two points on $\Lambda$ (namely, those with $\phi = \phi_*$) that would not be drawn right into the interior of $\Lambda$ (where $L < L_*$ in contradiction to the definition of $L_*$).

Remark (advanced). By linearizing the kinematic equations (2.1) and (2.3) at their stationary point $(r_*, \phi_*)$, we get the Jacobi matrix

$$J = \begin{pmatrix} 0 & \cos \phi_* \\ -r_*^{-2} \cos \phi_* & -r_*^{-1} \sin \phi_* \end{pmatrix} \begin{pmatrix} 0 & r_* \\ -1/r_* & -k/r_* \end{pmatrix}.$$ 

Its eigenvalues are

$$\lambda_{1,2} = \frac{k \pm \sqrt{k^2 - 4r_*^2}}{2r_*},$$

which are real with $\lambda_2 \leq \lambda_1 < 0$ if $2/\sqrt{5} \leq k < 1$ (Case A), and complex with negative real part if $0 < k < 2/\sqrt{5}$ (Case B).

Therefore, $Q_*$ is a hyperbolic stationary point, and the Hartman–Grobman theorem (see Hartman 1982, Thm. IX.7.1) shows that the motion looks, in the vicinity of that point, topologically like the solutions of the linear system $\dot{\eta} = J\eta$. In Case A, the stationary point is a node (improper if $k = 2/\sqrt{5}$) and, hence, the apparent motion of $Q$ does not spiral around $Q_*$ as $Q \to Q_*$. In Case B, the stationary point is a spiral point: then, because of

$$J_{12} - J_{21} = r_* + r_*^{-1} > 0,$$

the apparent motion of $Q$ spirals clockwise around $Q_*$ as $Q \to Q_*$; see Fig. 4 for the choice $k = 0.6$. Hathaway (1921, p. 281) erroneously alleged that the apparent path of the dog would always be a spiral converging to $Q_*$, for all $0 < k < 1$.

References