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A basic norm equivalence for the theory of multilevel methods

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Summary. Subspace decompositions of finite element spaces based on L_2-like orthogonal projections play an important role for the construction and analysis of multigrid like iterative methods. Recently several authors have proved the equivalence of the associated discrete norms with the H^1-norm. The present paper gives an elementary, self-contained derivation of this result which is based on the use of K-functionals known from the theory of interpolation spaces.

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1. Introduction

Stable subspace decompositions of finite element spaces have developed into a decisive tool for the construction and analysis of fast solvers for elliptic partial differential equations. For domain decomposition methods, this is a long accepted fact; we refer to [13], for example. Now we learn that the on the first sight completely different multigrid methods can also be interpreted and analyzed in this sense; they are multiplicative subspace correction methods [23]. This approach complements the classical interpretation which is reflected in Hackbusch's convergence theory [14], especially as it concerns nonuniformly refined meshes.

The analysis of multigrid methods as subspace correction methods began with the work of Bank and Dupont [3] and of Braess [6]. Later on, Yserentant developed a conceptionally very simple additive method of this type, which is especially well-suited to nonuniform grids, the hierarchical basis solver [24]. The multiplicative version of this method is the hierarchical basis multigrid method [4]. For a survey, we refer to [26].

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Given a sequence $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots$ of successively refined finite element spaces, the hierarchical basis methods are based on the splitting

$$u = I_0 u + \sum_{k=1}^{j} (I_k u - I_{k-1} u) \tag{1.1}$$

of the functions u in a final space $\mathcal{S} = \mathcal{S}_j$. Here, $I_k u \in \mathcal{S}_k$ denotes the finite element interpolant of u and j is the number of refinement levels. Bramble, Pasciak, and Xu [8, 21] had the idea to replace the decomposition (1.1) by a decomposition

$$u = Q_0 u + \sum_{k=1}^{j} (Q_k u - Q_{k-1} u) \tag{1.2}$$

with L_2-like projections Q_k onto $\mathcal{S}_k$. This was the break-through, which allowed the construction of additive methods [8, 21, 25] working for three- as well as for two space dimensions, and which allowed the analysis of classical multigrid methods for nonuniformly refined grids [9, 7, 22, 23].

With $\|\cdot\|_0$ a (weighted) L_2-norm and $\|\cdot\|$ the H^1-like energy norm induced by the second order boundary value problem under consideration, all these theories are based on the equivalence of the energy norm on $\mathcal{S} = \mathcal{S}_j$ with a discrete norm given by

$$|||u|||^2 = \|Q_0 u\|^2 + \sum_{k=1}^{j} 4^k \|Q_k u - Q_{k-1} u\|_0^2. \tag{1.3}$$

For uniformly refined grids and for the special class of nonuniform triangulations also considered in this paper, Bramble, Pasciak, and Xu [8] proved that

$$\frac{1}{K_1^* j} |||u|||^2 \leqq \|u\|^2 \leqq K_2^* j \, |||u|||^2. \tag{1.4}$$

These bounds are already very favorable. For quasiuniform grids of gridsize $\sim h$, the number j of refinement levels behaves like $|\log h|$. For H^2-regular problems, Bramble, Pasciak, and Xu could improve the lower bound to

$$|||u|||^2 \leqq K_1 \|u\|^2. \tag{1.5}$$

Yserentant [25] generalized (1.4) to the more flexible grids popularized by Bank's finite element package PLTMG [2] and carefully analyzed the influence of nonuniform initial triangulations. Later, Zhang [28] was able to improve the upper bound in (1.4) to

$$\|u\|^2 \leqq K_2 \, |||u|||^2. \tag{1.6}$$

Without assuming regularity properties of the boundary value problem under consideration, Oswald [18, 19] and Dahmen and Kunoth [12] proved the equivalence

$$\frac{1}{K_1}|||u|||^2 \leqq \|u\|^2 \leqq K_2|||u|||^2 \tag{1.7}$$

of the norm (1.3) to the energy norm with constants independent of the number j of refinement levels. Similar results have recently been derived by Xu [23] and by Zhang [28] and, for the lower bound, by Bramble and Pasciak [7]. The norm equivalence (1.7) transfers to nonuniformly refined grids. This is demonstrated by Dahmen and Kunoth [12] and follows also from considerations in [8] and [7].

The norm equivalence (1.7) immediately implies that the additive nodal basis preconditioner of Bramble et al. [8, 21, 25] reaches an optimal complexity, independent of the H^2- or even the $H^{1+\alpha}$-regularity of the boundary value problem under consideration and independent of the quasiuniformity of the family of grids. Together with a Cauchy-Schwarz type inequality as derived in Sect. 3 below, one can prove that this property transfers also to classical multigrid methods. This is shown in [23]. An optimal convergence result of this type has first been proved by Bramble and Pasciak [7]. Besides, the norm equivalence (1.7) represents a solid foundation for the construction and analysis of many new methods. A survey is given in [27].

The papers of Oswald and of Dahmen and Kunoth, respectively, are based on the fact that the Sobolev-space $H^1(\Omega)$ coincides with certain Besov-spaces and that the corresponding norms are equivalent. In addition, Oswald and Dahmen and Kunoth utilize that the approximation properties of L_2-like projections onto finite element spaces can be described by the second-order moduli of smoothness involved in the definition of the Besov-spaces. These tools have a long history and are well-established in approximation theory. On the other hand, the continuous discussion of the norm equivalence (1.7) among numerical mathematicians showed us that there is a strong need for a simpler, elementary and self-contained presentation of the topic which we want to give with this paper.

The paper is organized as follows. In Sect. 2 we introduce the finite element spaces and give a formal definition of the norms above. In Sect. 3, some technical results are derived. The most essential is a generalized Cauchy-Schwarz inequality which has first been stated by Xu (see [23]) and which is a modification of a result of Yserentant [24]. Similar estimates have been proven in [28] and in [7]. This inequality allows the proof of upper estimates like (1.6) for rather general subspace decompositions. In Sect. 4, this result is utilized to derive the upper bound (1.6). The next three sections treat the lower bound in (1.7). The crucial point is that the norm (1.3) changes, if the spaces $\mathcal{S}_k$ are replaced by subspaces. Following ideas ob Bramble, Pasciak, and Xu [8, 7], we show in Sect. 5 that it is sufficient to consider the uniformly refined case. The boundary values represent a special problem. In the context of partial differential equations, often one considers spaces of functions which vanish on certain parts of the boundary of the domain. The techniques in Sect. 7 cannot directly be applied to this situation. Therefore, in Sect. 6, we show that it is sufficient to consider the case that no boundary values are prescribed. This result is a special case of a theorem proven by Oswald [18].

Section 7 is the heart of our paper. Utilizing K-functionals instead of moduli of smoothness, we derive the lower bound in (1.7) for the remaining case of uniformly refined grids and free boundary values. Together with the results of the preceding sections, the desired norm equivalence follows.

Finally, in Sect. 8, stable local decompositions of finite element functions are constructed. They consist of finite element functions which vanish outside the refinement regions. Such decompositions are used in the analysis of multigrid methods with local smoothing procedures and of their additive counterparts.

2. The finite element spaces and norms

Although all our considerations are dimension independent or can easily be generalized to the three-dimensional case, for the ease of presentation we restrict out attention to two space dimensions.

Let $\bar{\Omega} \subseteq \mathbb{R}^2$ be a bounded polygonal domain without slits and let Γ be possibly empty subset of the boundary of Ω. By a *conforming* triangulation $\mathcal{T}$ of $\bar{\Omega}$, we mean a set of triangles such that the intersection of two such triangles is either empty or consists of a common edge or a common vertex. Here we start with an intentionally coarse conforming triangulation $\mathcal{T}_0$ of $\bar{\Omega}$. We assume that the boundary piece Γ is composed of edges of triangles in $\mathcal{T}_0$.

With the triangles $T \in \mathcal{T}_0$ we associate weights $\omega(T) > 0$. Similar to [25], we utilize the inner products

(2.1) $$(u, v)_G = \sum_{T \in \mathcal{T}_0} \frac{\omega(T)}{area(T)} \int_{G \cap T} uv\,dx$$

on $L_2(G)$ and

(2.2) $$D(u, v)|_G = \sum_{T \in \mathcal{T}_0} \omega(T) \sum_{i=1}^{2} \int_{G \cap T} D_i u D_i v\,dx$$

on $H^1(G)$ for subsets G of $\bar{\Omega}$. They induce the weighted L_2-norm given by

(2.3) $$\|u\|_{0;G}^2 = (u, u)_G$$

and the H^1-like seminorm given by

(2.4) $$|u|_{1;G}^2 = D(u, u)|_G.$$

For $G = \bar{\Omega}$, we omit the subscript G. In the application to elliptic boundary value problems, the weights $\omega(T)$ have the task to cover jumps of the coefficient functions across the boundaries of the triangles in the initial triangulation. In this paper (except for Sect. 7), constants are assumed to be independent of the $\omega(T)$. Note that for the weights $\omega(T) = 1$, (2.4) is the usual H^1-seminorm. (2.4) serves as a standardized energy norm. The task of the weights $1/area(T)$ in the L_2-like inner product (2.1) is to make the estimates independent of the sizes of the triangles in the initial triangulation. In the three-dimensional case, these factors have to be replaced by the factors $1/diam(T)^2$.

The triangulation $\mathcal{T}_0$ is refined several times, giving a family of nested, possibly *nonconforming* triangulations $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ A triangle of $\mathcal{T}_{k+1}$ is either a triangle of $\mathcal{T}_k$ or is generated by subdividing of triangle of $\mathcal{T}_k$ into four congruent subtriangles. The triangles of $\mathcal{T}_0$ are *level* 0 *elements*, and the triangles created by the refinement of level $k-1$ elements are *level k elements*. The mesh $\mathcal{T}_k$ may contain unrefined triangles from all lower levels, and thus it may be a highly nonuniform mesh.

We require that the level of two triangles in a triangulation $\mathcal{T}_k$, which have at least one common point, differs at most by one. This rule forces a patchwise refinement which means a certain restriction. The resulting meshes are sufficiently general to cover any desired type of singularity, but somewhat less flexible than the families of triangulations considered in [4] or [25], for example.

Finally, we demand that only triangles of the level k are refined in the transition from $\mathcal{T}_k$ to $\mathcal{T}_{k+1}$. This rule means that, for a given initial triangulation $\mathcal{T}_0$ and for a given final triangulation $\mathcal{T}_j$, the intermediate triangulations $\mathcal{T}_1, \ldots, \mathcal{T}_{j-1}$ can be reconstructed uniquely.

Corresponding to the triangulations $\mathcal{T}_k$ we have finite element spaces $\mathcal{S}_k$. $\mathcal{S}_k$ consists of all functions which are continuous on $\bar{\Omega}$ and linear on the triangles in $\mathcal{T}_k$ and which vanish on Γ. By construction, $\mathcal{S}_k$ is subspace of $\mathcal{S}_l$ for $k \leq l$.

The vertices of the triangles in $\mathcal{T}_k$ are called *nodes*. We distinguish the set $\mathcal{N}_k$ of the *free nodes* and the remaining set of the *nonfree nodes*. A function $u \in \mathcal{S}_k$ is determined by its values at the free nodes $x \in \mathcal{N}_k$, the values of u at the free nodes can be given arbitrarily. Under the given assumptions, nonfree nodes are always midpoints of edges connecting two free nodes. The values at the nonfree nodes can be computed by linear interpolation along these edges from the values at the corresponding neighbored free nodes.

In addition to the family of triangulations above, we need the uniformly refined family of triangulations $\tilde{\mathcal{T}}_0, \tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \ldots$ of $\bar{\Omega}$. $\tilde{\mathcal{T}}_0$ is the given initial triangulation $\mathcal{T}_0$. The triangles in $\tilde{\mathcal{T}}_{k+1}$ are generated by subdividing *all* triangles in $\tilde{\mathcal{T}}_k$ into four congruent subtriangles.

The space $\tilde{\mathcal{S}}_k$ consists of all functions which are continuous on $\bar{\Omega}$ and linear on the triangles in $\tilde{\mathcal{T}}_k$. The functions in $\tilde{\mathcal{S}}_k$ are given by their values at the nodes in $\tilde{\mathcal{N}}_k$ which are the vertices of the triangles in $\tilde{\mathcal{T}}_k$. The space $\mathcal{S}'_k$ consists of all those functions in $\tilde{\mathcal{S}}_k$ which vanish on the boundary piece Γ.

An advantage of our construction is that the spaces $\mathcal{S}_k$ are true subspaces of the spaces $\mathcal{S}'_k$ and $\tilde{\mathcal{S}}_k$ which simplifies our proofs considerably.

The L_2-like projections $Q_k: L_2 \to \mathcal{S}_k$, $Q'_k: L_2 \to \mathcal{S}'_k$, and $\tilde{Q}_k: L_2 \to \tilde{\mathcal{S}}_k$ are defined by

$$(Q_k u, v) = (u, v), \qquad v \in \mathcal{S}_k, \tag{2.5}$$

and correspondingly by

$$(Q'_k u, v'), \qquad v' \in \mathcal{S}'_k, \tag{2.6}$$

and by

$$(\tilde{Q}_k u, \tilde{v}) = (u, \tilde{v}), \qquad \tilde{v} \in \tilde{\mathcal{S}}_k. \tag{2.7}$$

For the rest of this paper we fix a final level j and denote the corresponding finite element spaces $\mathcal{S}_j$, $\mathcal{S}'_j$, and $\tilde{\mathcal{S}}_j$ by $\mathcal{S}$, $\mathcal{S}'$, and $\tilde{\mathcal{S}}$, respectively. On $\mathcal{S}$,

$\mathcal{S}'$, and $\tilde{\mathcal{S}}$, we introduce three closely related seminorms. These seminorms are

(2.8) $$|Q_0 u|_1^2 + \sum_{k=1}^{j} 4^k \|Q_k u - Q_{k-1} u\|_0^2,$$

on the finite element space $\mathcal{S} = \mathcal{S}_j$,

(2.9) $$|Q'_0 u|_1^2 + \sum_{k=1}^{j} 4^k \|Q'_k u - Q'_{k-1} u\|_0^2$$

on the finite element space $\mathcal{S}' = \mathcal{S}'_j$, and finally

(2.10) $$|\tilde{Q}_0 u|_1^2 + \sum_{k=1}^{j} 4^k \|\tilde{Q}_k u - \tilde{Q}_{k-1} u\|_0^2$$

on the finite element space $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}_j$. Our aim is the comparison of these seminorms by each other and primarily their comparison with the weighted H^1-seminorm

(2.11) $$|u|_1^2 = D(u, u).$$

As it will turn out, all these seminorms are equivalent on $\mathcal{S}$.

We end with some remarks concerning the coefficients 4^k. These factors are not dimension dependent, but reflect the refinement strategy. For example, they have to be replaced by the factors 9^k, if one reduces the diameters of the triangles by the factor 3 instead by the factor 2 in the transition from one level to the next.

3. Some technical results

Before we start with the comparison theorems, we derive some basic properties of the finite element spaces. We begin with an inverse inequality.

Lemma 3.1. *For all functions $v \in \tilde{\mathcal{S}}_k$ and all triangles $T \in \tilde{\mathcal{T}}_k$,*

(3.1) $$|v|_{1;T}^2 \leqq C_0 4^k \|v\|_{0;T}^2.$$

The constant C_0 depends only on the shape regularity of the triangles in the initial triangulation.

We omit the proof which is standard and remark only that the factor 4^k enters because we normalized the L_2-like norm (2.3) by the areas of the triangles in the initial triangulation.

Several times we will cut off finite element functions. Under this operation, the L_2-like norm (2.3) cannot increase very much.

Lemma 3.2. *Let Δ be a triangle which is a subset of a triangle in $\mathcal{T}_0$. Let $\bar{v}$ be a linear function taking the same values as v at one or two vertices of Δ and the value 0 at the remaining vertices of Δ. Then*

(3.2) $$\|\bar{v}\|_{0;\Delta}^2 \leqq \tfrac{3}{2} \|v\|_{0;\Delta}^2.$$

Proof. If a linear function w takes the values w_1, w_2 and w_3 at the vertices of Δ, then

$$\int_\Delta |w(x)|^2\,dx = \frac{area(\Delta)}{12}\,(w_1^2+w_2^2+w_3^2+(w_1+w_2+w_3)^2).$$

Thus the proposition follows from the estimates

$$v_1^2+v_2^2+(v_1+v_2)^2 \leqq \tfrac{3}{2}\,(v_1^2+v_2^2+v_3^2+(v_1+v_2+v_3)^2),$$

corresponding to the case that one value is set to zero, and

$$v_1^2+v_1^2 \leqq \tfrac{3}{2}\,(v_1^2+v_2^2+v_3^2+(v_1+v_2+v_3)^2)$$

corresponding to the other case. These estimates cannot be improved. □

The main result of this section is a Cauchy-Schwarz type inequality which has been stated by Xu (see [23]), but which has essentially already been proven in [24]. Related results can be found in [7] and in [28].

Lemma 3.3. *There is a constant C depending only on the shape regularity of the triangles under consideration with*

$$D(v,w)|_T \leqq C\left(\frac{1}{\sqrt{2}}\right)^{l-k} |v|_{1;T}\, 2^l\, \|w\|_{0;T} \tag{3.3}$$

for all triangles $T\in\tilde{\mathscr{T}}_k$ and for all functions $v\in\tilde{\mathscr{S}}_k$, $w\in\tilde{\mathscr{S}}_l$, $l>k$.

Proof. For $l=k+1$, one can apply Lemma 3.1 to w and gets (3.3) with the constant $C=\sqrt{2C_0}$. For $l\geqq k+2$, let $w_0\in\tilde{\mathscr{S}}_l$ be given by

$$w_0(x)=\begin{cases} w(x), & x\in\tilde{\mathscr{N}}_l\cap\partial T\\ 0, & x\in\tilde{\mathscr{N}}_l\setminus\partial T\end{cases}$$

Then, with $w_1=w-w_0$, the inner product to be estimated can be written as

$$D(v,w)|_T = D(v,w_0)|_T + D(v,w_1)|_T.$$

The essential point is that w_1 vanishes on the boundary of T. As v is linear on T, therefore we obtain, by partial integration,

$$D(v,w_1)|_T=0.$$

The function w_0 vanishes outside a boundary strip S of T with

$$\frac{area(S)}{area(T)} = 1-(1-3(\tfrac{1}{2})^{l-k})^2 \leqq 6(\tfrac{1}{2})^{l-k}.$$

As the first order derivatives of v are constant on T,

$$D(v,w_0)|_T \leqq |v|_{1;S}|w_0|_{1;S} \leqq \sqrt{6}\left(\frac{1}{\sqrt{2}}\right)^{l-k} |v|_{1;T}|w_0|_{1;S}$$

follows. Lemma 3.1 and Lemma 3.2 yield

$$|w_0|_{1;S}^2 \leqq C_0 4^l \|w_0\|_{0;S}^2 \leqq \tfrac{3}{2} C_0 4^l \|w\|_{0;T}^2.$$

This proves the proposition with the constant $C = 3\sqrt{C_0}$. □

The local estimate (3.3) implies the global estimate

$$D(v, w) \leqq C\left(\frac{1}{\sqrt{2}}\right)^{l-k} |v|_1 2^l \|w\|_0 \tag{3.4}$$

for the functions $v \in \tilde{\mathscr{S}}_k$, $w \in \tilde{\mathscr{S}}_l$ and $l > k$. In [24], in the next section, and probably in most other interesting cases, (3.4) is applied to functions $w \in \tilde{\mathscr{S}}_l$ for which $2^l \|w\|_0$ can be estimated by $|w|_1$. One gets the strengthened Cauchy-Schwarz inequality

$$D(v, w) \leqq \hat{C}\left(\frac{1}{\sqrt{2}}\right)^{l-k} |v|_1 |w|_1.$$

This fact justifies the name Cauchy-Schwarz type inequality for (3.3) and (3.4).

We remark that, utilizing the same idea, estimates corresponding to (3.3) and (3.4) can be proven for any bilinear form

$$a(u, v) = \sum_{i,k=1}^{2} \int_\Omega a_{ik} D_i u D_k v \, dx \tag{3.5}$$

with coefficient functions a_{ik} which are continuously differentiable on the triangles in the initial triangulation. The only difference to the proof of (3.3) is that one obtains

$$a(v, w_1)|_T \leqq M^* |v|_{1;T} \|w_1\|_{0;T} \leqq \sqrt{\tfrac{3}{2}} M^* |v|_{1;T} \|w\|_{0;T} \tag{3.6}$$

instead of $D(v, w_1)|_T = 0$. The generalization to higher order functions is also possible; see [27].

The estimate (3.4) is the basis for the proof of the last result in this section which has a rather general range of application.

Lemma 3.4. *Assume that a function $u \in \tilde{\mathscr{S}}$ has the decomposition*

$$u = u_0 + \sum_{k=1}^{j} u_k, \qquad u_k \in \tilde{\mathscr{S}}_k. \tag{3.7}$$

Then there exists a constant K depending only on the shape regularity of the triangles in the initial triangulation, but not on j, with

$$|u|_1^2 \leqq K\left\{|u_0|_1^2 + \sum_{k=1}^{j} 4^k \|u_k\|_0^2\right\}. \tag{3.8}$$

Proof. Let

$$\xi_0 = |u_0|_1, \qquad \xi_k = 2^k \|u_k\|_0 \quad (k \geqq 1).$$

Then by (3.4) and the inverse estimate (3.1),

$$|u|_1^2 = \left| \sum_{k=0}^{j} u_k \right|_1^2 = \sum_{k,l=0}^{j} D(u_k, u_l) \leqq \tilde{c} \sum_{k,l=0}^{j} \left(\frac{1}{\sqrt{2}} \right)^{|k-l|} \xi_k \xi_l .$$

As the largest eigenvalue of the matrix with the entries

$$a_{kl} = \left(\frac{1}{\sqrt{2}} \right)^{|k-l|}$$

is bounded by

$$\max_{k=0,\ldots,j} \sum_{l=0}^{j} |a_{kl}| \leqq 1 + 2 \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{2}} \right)^k = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

one obtains

$$|u|_1^2 \leqq \tilde{c} \frac{\sqrt{2}+1}{\sqrt{2}-1} \sum_{k=0}^{j} \xi_k^2$$

which is the proposition. □

4. The upper bound

As an immediate consequence of Lemma 3.4, we can derive Zhang's upper bound (1.6) and can estimate the weighted H^1-seminorm (2.11) on the fixed final space $\mathscr{S} = \mathscr{S}_j$ by the seminorm (2.8).

Theorem 4.1. *There is a constant K_2, which depends only on the shape regularity of the triangles in the initial triangulation, with*

$$|u|_1^2 \leqq K_2 \left\{ |Q_0 u|_1^2 + \sum_{k=1}^{j} 4^k \|Q_k u - Q_{k-1} u\|_0^2 \right\} \tag{4.1}$$

for all functions $u \in \mathscr{S}$.

Proof. Apply Lemma 3.4 to the decomposition

$$u = Q_j u = Q_0 u + \sum_{k=1}^{j} (Q_k u - Q_{k-1} u). \quad \square$$

It should be noted that (4.1) is a local estimate. Neither the size of the triangles in the initial triangulation nor the weights $\omega(T)$ enter into the constant.

Of course, corresponding estimates can be proven for the spaces $\mathscr{S}'$ and $\tilde{\mathscr{S}}$ with the seminorm (2.8) replaced by the seminorms (2.9) and (2.10), respectively.

5. The reduction to the case of uniformly refined meshes

The proof, that the seminorm (2.8) can conversely be estimated by the weighted H^1-seminorm (2.11), is a much more delicate task. As mentioned in the introduction, this proof consists of several steps. Following ideas of Bramble, Pasciak and Xu [8, 7], we show in this section that it is sufficient to analyze the case of a uniform refinement.

As $\mathscr{S}_k$ is a subset of the space $\mathscr{S}'_k$ arising from a uniform refinement, it is trivial that

$$\|u - Q'_k u\|_0 \leqq \|u - Q_k u\|_0 \tag{5.1}$$

for all $u \in L_2(\Omega)$. The key observation in this section, going back to [8] and [7], is that, for functions u in the finite element space $\mathscr{S} = \mathscr{S}_j$, the right-hand side of (5.1) can conversely be estimated by the left hand side. At this place the special patchwise refinement structure enters.

Lemma 5.1. *For $u \in \mathscr{S}$ and $k = 0, 1, \ldots, j$,*

$$\|u - Q_k u\|_0 \leqq \sqrt{\tfrac{3}{2}}\, \|u - Q'_k u\|_0 . \tag{5.2}$$

Proof. We define the subset $\bar{\Omega}_k$ of $\bar{\Omega}$ as the union of all triangles in $\mathscr{T}_j$ of a level less than k. Of course, it is possible that $\bar{\Omega}_k$ is the empty set. Let the mapping $R_k \colon \mathscr{S}' \to \mathscr{S}$ be given by

$$(R_k v)(x) = \begin{cases} v(x), & x \in \mathscr{N}_j \setminus \bar{\Omega}_k \\ 0, & x \in \mathscr{N}_j \cap \bar{\Omega}_k . \end{cases}$$

As, by assumption, triangles in $\mathscr{T}_j$ of a level greater than k cannot intersect $\bar{\Omega}_k$, R_k maps $\mathscr{S}'_k$ into $\mathscr{S}_k$. For the same reason and by Lemma 3.2, the estimate

$$\|R_k v\|_0^2 \leqq \tfrac{3}{2} \|v\|_0^2 \tag{5.3}$$

holds for the linear combinations v of functions in $\mathscr{S}$ and $\mathscr{S}'_k$.

Introducing $v = u - I_k u$, with $I_k u \in \mathscr{S}_k$ the finite element interpolation of $u \in \mathscr{S}$, we obtain

$$\|u - Q_k u\|_0 = \|v - Q_k v\|_0 \leqq \|v - R_k Q'_k v\|_0 .$$

As v vanishes on $\bar{\Omega}_k$, we have $v = R_k v$. Therefore

$$\|u - Q_k u\|_0 \leqq \|R_k(v - Q'_k v)\|_0 = \|R_k(u - Q'_k u)\|_0 .$$

The proposition now follows from (5.3). □

Using Lemma 5.1, the seminorm (2.8) of a function $u \in \mathscr{S}$ can be estimated by its seminorm (2.9).

Theorem 5.2. *For $u \in \mathscr{S}$,*

$$|Q_0 u|_1^2 + \sum_{k=1}^{j} 4^k \|Q_k u - Q_{k-1} u\|_0^2 \leqq 2 \left\{ |Q'_0 u|_1^2 + \sum_{k=1}^{j} 4^k \|Q'_k u - Q'_{k-1} u\|_0^2 \right\}. \tag{5.4}$$

Proof. By Lemma 5.1 we have

$$\sum_{k=1}^{j} 4^k \|Q_k u - Q_{k-1} u\|_0^2 \leqq \sum_{k=1}^{j} 4^k \|u - Q_{k-1} u\|_0^2$$
$$\leqq \tfrac{3}{2} \sum_{k=1}^{j} 4^k \|u - Q'_{k-1} u\|_0^2$$
$$\leqq \tfrac{3}{2} \sum_{k=1}^{j} 4^k \sum_{l=k}^{j} \|Q'_l u - Q'_{l-1} u\|_0^2$$

Changing the order of summation gives

$$\sum_{k=1}^{j} 4^k \|Q_k u - Q_{k-1} u\|_0^2 \leqq \tfrac{3}{2} \sum_{l=1}^{j} \left(\sum_{k=1}^{l} 4^k\right) \|Q'_l u - Q'_{l-1} u\|_0^2$$
$$\leqq \tfrac{3}{2} \sum_{l=1}^{j} \tfrac{4}{3} 4^l \|Q'_l u - Q'_{l-1} u\|_0^2.$$

As $\mathscr{S}_0 = \mathscr{S}'_0$ and therefore $Q_0 u = Q'_0 u$, the proposition follows. □

6. The reduction to the case that no boundary values are prescribed

In this section we show that it is sufficient to estimate the seminorm (2.10) instead of the seminorm (2.9). This result is essentially due to Oswald [18]. Here it is formulated and proved in a different form.

The constants in this section depend on the jumps of the coefficients $\omega(T)$ near the boundary Γ. We introduce the constant

$$\sigma(\Gamma) = \max\left\{\frac{\omega(T_1)}{\omega(T_2)} \,\middle|\, T_i \in \mathscr{T}_0,\ T_1 \cap T_2 \cap \Gamma \neq \emptyset\right\}. \tag{6.1}$$

Our aim is the proof of the following

Theorem 6.1. *If there exists a constant $\tilde{K}_1$ independent of j with*

$$|\tilde{Q}_0 u|_1^2 + \sum_{k=1}^{j} 4^k \|\tilde{Q}_k u - \tilde{Q}_{k-1} u\|_0^2 \leqq \tilde{K}_1 |u|_1^2 \tag{6.2}$$

for all functions $u \in \mathscr{S}' = \mathscr{S}'_j$, then there exists also a constant K'_1 independent of j with

$$|Q'_0 u|_1^2 + \sum_{k=1}^{j} 4^k \|Q'_k u - Q'_{k-1} u\|_0^2 \leqq K'_1 \sigma(\Gamma) |u|_1^2 \tag{6.3}$$

for all $u \in \mathscr{S}'$.

The proof of this theorem is based on the

Lemma 6.2. *Given a decomposition*

$$u = u_0 + \sum_{k=1}^{j} u_k, \qquad u_k \in \tilde{\mathscr{S}}_k, \tag{6.4}$$

of a function $u \in \mathscr{S}'$, there exists a decomposition

$$u = u'_0 + \sum_{k=1}^{j} u'_k, \qquad u'_k \in \mathscr{S}'_k, \tag{6.5}$$

with

$$\sum_{k=1}^{j} 4^k \|u'_k\|_0^2 \leqq \hat{c}\,\sigma(\Gamma) \left\{ |u_0|_1^2 + \sum_{k=1}^{j} 4^k \|u_k\|_0^2 \right\} \tag{6.6}$$

where $\hat{c}$ is independent of j and of the $\omega(T)$.

Accepting this proposition for a while, we can give a

Proof of Theorem 6.1. Let $u \in \mathscr{S}'$. Since

$$u = \tilde{Q}_0 u + \sum_{k=1}^{j} (\tilde{Q}_k u - \tilde{Q}_{k-1} u)$$

and because of the assumption (6.2), there exists, by Lemma 6.2, a decomposition

$$u = u'_0 + \sum_{k=1}^{j} u'_k, \qquad u'_k \in \mathscr{S}'_k,$$

with

$$\sum_{k=1}^{j} 4^k \|u'_k\|_0^2 \leqq c_0\,\sigma(\Gamma) |u|_1^2. \tag{6.7}$$

The Cauchy-Schwarz inequality leads to

$$\sum_{k=1}^{j} 4^k \|Q'_k u - Q'_{k-1} u\|_0^2 = \sum_{l=1}^{j} \sum_{k=1}^{l} 4^k (Q'_k u - Q'_{k-1} u, Q'_k u'_l - Q'_{k-1} u'_l)$$

$$\leqq \left\{ \sum_{l=1}^{j} \sum_{k=1}^{l} 4^{2k-l} \|Q'_k u - Q'_{k-1} u\|_0^2 \right\}^{1/2} \left\{ \sum_{l=1}^{j} \sum_{k=1}^{l} 4^l \|Q'_k u'_l - Q'_{k-1} u'_l\|_0^2 \right\}^{1/2}.$$

A rearrangement of the first sum gives

$$\sum_{l=1}^{j} \sum_{k=1}^{l} 4^{2k-l} \|Q'_k u - Q'_{k-1} u\|_0^2 = \sum_{k=1}^{j} \sum_{l=k}^{j} 4^{2k-l} \|Q'_k u - Q'_{k-1} u\|_0^2$$

$$\leqq \tfrac{4}{3} \sum_{k=1}^{j} 4^k \|Q'_k u - Q'_{k-1} u\|_0^2.$$

For the second sum on the right-hand side one obtains

$$\sum_{l=1}^{j} \sum_{k=1}^{l} 4^l \|Q'_k u'_l - Q'_{k-1} u'_l\|_0^2 \leqq \sum_{l=1}^{j} 4^l \|u'_l\|_0^2.$$

Together this yields

$$\sum_{k=1}^{j} 4^k \|Q'_k u - Q'_{k-1} u\|_0^2 \leqq \tfrac{4}{3} \sum_{l=1}^{j} 4^l \|u'_l\|_0^2.$$

With (6.7) one obtains

$$\sum_{k=1}^{j} 4^k \|Q'_k u - Q'_{k-1} u\|_0^2 \leqq c_1 \sigma(\Gamma) |u|_1^2.$$

As by Lemma 3.4

$$|u - Q'_0 u|_1^2 = \left| \sum_{k-1}^{j} (Q'_k u - Q'_{k-1} u) \right|_1^2 \leqq c_2 \sum_{k=1}^{j} 4^k \|Q'_k u - Q'_{k-1} u\|_0^2,$$

$|u - Q'_0 u|_1^2$ and therefore also $|Q'_0 u|_1^2$ can be estimated by $|u|_1^2$, too. □

For the proof of the lemma, we split the functions $u_1, \ldots, u_j$ in (6.4) into sums

(6.8) $$u_k = v'_k + w_k, \qquad v'_k \in \mathscr{S}'_k,\ w_k \in \tilde{\mathscr{S}}_k,$$

where w_k is given by

(6.9) $$w_k(x) = \begin{cases} u_k(x), & x \in \tilde{\mathscr{N}}_k \cap \Gamma \\ 0, & x \in \tilde{\mathscr{N}}_k \setminus \Gamma. \end{cases}$$

We set

(6.10) $$w = u_0 + \sum_{k=1}^{j} w_k.$$

By definition, w coincides with u on the boundary piece Γ. As $u \in \mathscr{S}'$ vanishes on Γ, w vanishes there, too. Let

(6.11) $$u'_0 = \tilde{I}_0 w, \qquad w'_k = \tilde{I}_k w - \tilde{I}_{k-1} w \quad (k \geqq 1)$$

with the finite element interpolation operators $\tilde{I}_k$: $\tilde{\mathscr{S}} \to \tilde{\mathscr{S}}_k$. Then $u'_0 \in \mathscr{S}'_0$, $w'_k \in \mathscr{S}'_k$ and $w'_k(x) = 0$ for $x \in \tilde{\mathscr{N}}_{k-1}$.

(6.12) $$w = u'_0 + \sum_{k=1}^{j} w'_k$$

is the hierarchical decomposition of w. For $k \geqq 1$, let

(6.13) $$u'_k = v'_k + w'_k.$$

The desired decomposition (6.5) of u is

$$u = u_0' + \sum_{k=1}^{j} u_k'. \tag{6.14}$$

The backbone of the proof of the lemma is the estimate

$$\sum_{k=1}^{j} 4^k \|w_k'\|_0^2 \leqq c_3 \sigma(\Gamma) |w|_1^2. \tag{6.15}$$

We postpone the proof of (6.15) and continue the proof of our lemma. Utilizing the representation (6.10) of w and Lemma 3.4, (6.15) leads to

$$\sum_{k=1}^{j} 4^k \|w_k'\|_0^2 \leqq c_3 \sigma(\Gamma) \left| u_0 + \sum_{k=1}^{j} w_k \right|_1^2 \leqq c_4 \sigma(\Gamma) \left\{ |u_0|_1^2 + \sum_{k=1}^{j} 4^k \|w_k\|_0^2 \right\}.$$

By Lemma 3.2,

$$\|w_k\|_0^2 \leqq \tfrac{3}{2} \|u_k\|_0^2.$$

Therefore

$$\sum_{k=1}^{j} 4^k \|w_k'\|_0^2 \leqq c_5 \sigma(\Gamma) \left\{ |u_0|_1^2 + \sum_{k=1}^{j} 4^k \|u_k\|_0^2 \right\}.$$

Using $u_k' = v_k' + w_k'$, and as also

$$\|v_k'\|_0^2 \leqq \tfrac{3}{2} \|u_k\|_0^2,$$

we obtain the final estimate

$$\sum_{k=1}^{j} 4^k \|u_k'\|_0^2 \leqq \hat{c} \sigma(\Gamma) \left\{ |u_0|_1^2 + \sum_{k=1}^{j} 4^k \|u_k\|_0^2 \right\}.$$

This proves the lemma.

For the remaining proof of (6.15), first we remark that, for all linear functions v and all triangles $T \in \tilde{\mathscr{T}}_k$ with $T \subseteq \hat{T}$, $\hat{T} \in \mathscr{T}_0$,

$$\frac{1}{4} \|v\|_{0;T}^2 \leqq \omega(\hat{T}) \frac{4^{-k}}{12} \sum_{x \in \tilde{\mathscr{N}}_k \cap T} |v(x)|^2 \leqq \|v\|_{0;T}^2. \tag{6.16}$$

Let $\tilde{\mathscr{T}}_k^*$ be the set consisting of all triangles in $\tilde{\mathscr{T}}_k$ which intersect Γ along a whole edge. Utilizing (6.16) and the special structure of the functions w_k', one recognizes that

$$\|w_k'\|_0^2 \leqq \gamma \sigma(\Gamma) \sum_{T \in \tilde{\mathscr{T}}_{k-1}^*} \|w_k'\|_{0;T}^2 \tag{6.17}$$

where γ is a simple constant which depends only on the maximum number of triangles in $\mathcal{T}_0$ having a common vertex on Γ. Therefore it is sufficient to show the estimate

$$\sum_{k=1}^{j} 4^k \sum_{T\in\mathcal{T}_{k-1}^*,\, T\subseteq\hat{T}} \|w_k'\|_{0;T}^2 \leqq c_6 |w|_{1;\hat{T}}^2 \tag{6.18}$$

for a given fixed triangle $\hat{T}\in\mathcal{T}_0^*$.

Without restriction we can assume that this triangle $\hat{T}\in\mathcal{T}_0^*$ has the vertices (0, 0), (0, 1) and (a, b), $a>0$, and that the intersection of $\hat{T}$ with Γ is the line

$$x_2=\frac{b}{a}x_1, \qquad 0\leqq x_1\leqq a,$$

connecting the vertices (0, 0) and (a, b). We subdivide $\hat{T}$ into the parallel strips

$$G_k=\left\{(x_1,x_2)\in\hat{T}\,\middle|\,\frac{b}{a}x_1+2^{-k}\leqq x_2\leqq\frac{b}{a}x_1+2^{-(k-1)}\right\}$$

for $k=1,\dots,j$ and a remaining strip

$$G_{j+1}=\left\{(x_1,x_2)\in\hat{T}\,\middle|\,\frac{b}{a}x_1\leqq x_2\leqq\frac{b}{a}x_1+2^{-j}\right\}.$$

We consider a fixed triangle $T\in\mathcal{T}_{k-1}^*$ with vertices P_1, P_2, P_3 where we assume that the line connecting P_2 with P_3 is the intersection of T with Γ. Let P_4 be the midpoint of the line connecting P_1 with P_2, P_5 be the midpoint of the line connecting P_2 and P_3 and P_6 be the midpoint of the line connecting P_3 with P_1. w_k' is linear on the four subtriangles in $\mathcal{T}_k$ the union of which is T, and w itself is linear on the triangle T' with the vertices P_1, P_4 and P_6. One has

$$w_k'(P_1)=w_k'(P_2)=w_k'P_3)=w_k'(P_5)=0$$

and

$$w_k'(P_i)=w(P_i)-\tfrac{1}{2}w(P_1), \qquad i=4,6.$$

A study of the associated quadratic forms shows that

$$\|w_k'\|_{0;T}^2\leqq 10\,\|w\|_{0;T'}^2=10\,\|w\|_{0;T\cap G_k}^2.$$

Therefore

$$\sum_{k=1}^{j} 4^k \sum_{T\in\mathcal{T}_{k-1}^*,\, T\subseteq\hat{T}} \|w_k'\|_{0;T}^2\leqq 10\sum_{k=1}^{j} 4^k\,\|w\|_{0;G_k}^2. \tag{6.19}$$

Because of

$$\frac{area(\hat{T})}{\omega(\hat{T})}\sum_{k=1}^{j} 4^k\,\|w\|_{0;G_k}^2\leqq 4\int_{\hat{T}}\left(x_2-\frac{b}{a}x_1\right)^{-2}|w(x)|^2\,dx$$
$$=4\int_0^a\left\{\int_{\frac{b}{a}x_1}^{\frac{b-1}{a}x_1+1}\left(x_2-\frac{b}{a}x_1\right)^{-2}|w(x_1,x_2)|^2\,dx_2\right\}dx_1,$$

(6.15) now follows from the elementary estimate

$$\int_\alpha^\beta \frac{1}{(t-\alpha)^2} f(t)^2\,dt = \int_\alpha^\beta \left[\frac{1}{t-\alpha} - \frac{1}{\beta-\alpha}\right] 2f(t)f'(t)\,dt \leqq 4 \int_\alpha^\beta f'(t)^2\,dt$$

for continuous and piecewise linear functions $f\colon [\alpha, \beta] \to \mathbb{R}$ with $f(\alpha)=0$ and the fact that $area(\hat{T}) \sim 1$ for the special triangle $\hat{T}$ under consideration here.

7. The lower bound

There remains to estimate the discrete seminorm (2.10) by the H^1-like seminorm (2.11). This is the key result of Oswald's papers as well as of Dahmen's and Kunoth's paper. Here we present a strongly simplified proof which avoids the use of Besov-spaces. Instead of moduli of smoothness, we use the more flexible K-functionals.

We restrict out attention to the case

$$\omega(T) = 1, \qquad T \in \mathscr{T}_0. \tag{7.1}$$

In addition, we assume that the *initial* triangulation is quasiuniform. That means that there are positive constants α_0 and α_1 of order 1 and a meshsize H with

$$\alpha_0 H \leqq diam(T) \leqq \alpha_1 H, \qquad T \in \mathscr{T}_0. \tag{7.2}$$

As a change of the size of the domain Ω does not affect the ratio of the seminorms (2.10) and (2.11), we can assume

$$H = 1. \tag{7.3}$$

With (7.1) and (7.2), (7.3), the weighted L_2-norm (2.3) behaves like the usual L_2-norm $\|\cdot\|_{0,2;\Omega}$ without weights and the seminorm (2.11) is the $H^1(\Omega)$-seminorm $|\cdot|_{1,2;\Omega}$. At the end of this section, we give a short discussion of the general case.

For $u \in L_2(\Omega)$ and $t > 0$, we introduce the (slightly modified) K-functional

$$K(t, u) = \inf_{v \in H^2(\Omega)} \{\|u - v\|_{0,2}^2 + t^2 |v|_{2,2}^2\}^{1/2} \tag{7.4}$$

where $|\cdot|_{2,2} = |\cdot|_{2,2;\Omega}$ is the usual H^2-seminorm given by

$$|v|_{2,2}^2 = \sum_{m+n=2} \binom{2}{m} \int_\Omega |(D_1^m D_2^n v)(x)|^2\,dx. \tag{7.5}$$

K-functionals play a dominant role for the construction and analysis of interpolation spaces [16] and in abstract approximation theory; see [5] and [10], for example.

One can use the K-functional (7.4) to describe the approximation properties of the projections $\tilde{Q}_k\colon L_2(\omega) \to \mathscr{S}_k$.

Lemma 7.1. *There exists a constant* c_0 *such that*

(7.6) $$\|u-\tilde{Q}_k u\|_0 \leqq c_0 K(4^{-k}, u)$$

for all functions $u \in L_2(\Omega)$.

Proof. Let $\tilde{I}_k v \in \tilde{\mathscr{S}}_k$ be the interpolant of an arbitrary function $v \in H^2(\Omega)$. Then,

$$\|u-\tilde{Q}_k u\|_0 \leqq \|u-\tilde{I}_k v\|_0 \leqq \|u-v\|_0 + \|v-\tilde{I}_k v\|_0$$

holds. By the assumptions (7.1), (7.2), and (7.3),

$$\|u-\tilde{Q}_k u\|_0 \leqq c^* \{\|u-v\|_{0,2}^2 + \|v-\tilde{I}_k v\|_{0,2}^2\}^{1/2}$$

follows. As, by (7.2) and (7.3), the triangles in $\tilde{\mathscr{T}}_k$ have diameters $\sim 2^{-k}$, the finite element approximation theory (see [11] or [15]) states

$$\|v-\tilde{I}_k v\|_{0,2} \leqq \hat{c}(2^{-k})^2 |v|_{2,2}.$$

This yields

$$\|u-\tilde{Q}_k u\|_0 \leqq c_0 \{\|u-v\|_{0,2}^2 + (4^{-k})^2 |v|_{2,2}^2\}^{1/2}.$$

Taking the infimum over all $v \in H^2(\Omega)$, one obtains the proposition. □

Utilizing (7.6), we can estimate the critical part of the discrete seminorm (2.10) in terms of the K-functional.

Lemma 7.2. *For all* $u \in L_2(\Omega)$,

(7.7) $$\sum_{k=1}^{j} 4^k \|\tilde{Q}_k u - \tilde{Q}_{k-1} u\|_0^2 \leqq 4 c_0^2 \sum_{k=0}^{j-1} 4^k K(4^{-k}, u)^2.$$

Proof. (7.6) yields

$$\sum_{k=1}^{j} 4^k \|\tilde{Q}_k u - \tilde{Q}_{k-1} u\|_0^2 \leqq \sum_{k=1}^{j} 4^k \|u - \tilde{Q}_{k-1} u\|_0^2 \leqq \sum_{k=1}^{j} 4^k c_0^2 K(4^{-(k-1)}, u)^2. \quad \square$$

Next we show that, for $u \in \mathscr{S}'$, the right-hand side of (7.7) can be estimated by the H^1-seminorm of the function u. First, we have to consider functions which are defined on the whole $\mathbb{R}^2$.

Lemma 7.3. *For all* $u \in H^1(\mathbb{R}^1)$,

(7.8) $$\sum_{k=0}^{\infty} 4^k K(4^{-k}, u; \mathbb{R}^2)^2 \leqq \frac{2\pi}{3} |u|_{1,2;\mathbb{R}^2}^2.$$

Proof. Expressing the norms in terms of Fourier-transforms, one finds

$$K(t, u; \mathbb{R}^2)^2 = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{t^2 |\omega|^4}{1 + t^2 |\omega|^4} |\hat{u}(\omega)|^2 d\omega$$

and obtains

$$\sum_{k=0}^{\infty} 4^k K(4^{-k}, u; \mathbb{R}^2)^2 = \frac{1}{2\pi} \int_{\mathbb{R}^2} S(\omega)|\omega|^2 |\hat{u}(\omega)|^2 d\omega$$

where the kernel $S(\omega)$ is given by

$$S(\omega) = \sum_{k=0}^{\infty} 4^k \frac{4^{-2k}|\omega|^2}{1+4^{-2k}|\omega|^4}.$$

It satisfies the estimate

$$S(\omega) \leqq \frac{4}{3} \int_0^1 \frac{|\omega|^2}{1+s^2|\omega|^4} ds \leqq \frac{4}{3} \frac{\pi}{2}.$$

This yields the proposition

$$\sum_{k=0}^{\infty} 4^k K(4^{-k}, u; \mathbb{R}^2)^2 \leqq \frac{2\pi}{3} \frac{1}{2\pi} \int_{\mathbb{R}^2} |\omega|^2 |\hat{u}(\omega)|^2 d\omega. \quad \square$$

Now we are in the position to treat the case of our polygonal domain Ω. For this purpose, we need an extension operator

$$E: \mathscr{S}' \to H^2(\mathbb{R}^2) \tag{7.9}$$

which is bounded independent of the number of refinement levels and independent of the initial triangulation of the domain; for all $u \in \mathscr{S}'$, let

$$|Eu|_{1,2;\mathbb{R}^2}^2 \leqq c(\Omega)|u|_{1,2;\Omega}^2. \tag{7.10}$$

If Γ consists of the whole boundary of Ω, the trivial extension by zero satisfies (7.10) with $c(\Omega)=1$; for this special case, one could also allow domains with slits. The general case is more complicated, but as we have assumed that the domain Ω has a Lipschitz-boundary, one can construct an extension operator $E: H^1(\Omega) \to H^1(\mathbb{R}^2)$ which satisfies (7.10) for all $u \in H^1(\Omega)$; we refer to [20, 17], or [1]. Unfortunately, the constant $c(\Omega)$ can tend to infinity, if an exterior angle of Ω tends to zero, i.e. if Ω approaches a slit domain.

Utilizing (7.10) we can prove:

Lemma 7.4. *There exists a constant c_1 such that, for all $u \in \mathscr{S}'$,*

$$\sum_{k=0}^{\infty} 4^k K(4^{-k}, u; \Omega)^2 \leqq c_1 |u|_{1,2;\Omega}^2. \tag{7.11}$$

Proof. Using the trivial estimate

$$K(t, u; \Omega) \leqq K(t, Eu; \mathbb{R}^2),$$

one gets

$$\sum_{k=0}^{\infty} 4^k K(4^{-k}, u; \Omega)^2 \leqq \sum_{k=0}^{\infty} 4^k K(4^{-k}, Eu; \mathbb{R}^2)^2 \leqq \frac{2\pi}{3} |Eu|_{1,2;\mathbb{R}^2}^2.$$

With (7.10), the proposition follows. □

Combining (7.7) and (7.11), we end up with the estimate

$$\sum_{k=1}^{j} 4^k \|\tilde{Q}_k u - \tilde{Q}_{k-1} u\|_0^2 \leqq c_2 |u|_1^2, \quad u \in \mathcal{S}'. \tag{7.12}$$

For the functions u in the finite element space $\mathcal{S}'$, the representation

$$u - \tilde{Q}_0 u = \sum_{k=1}^{j} (\tilde{Q}_k u - \tilde{Q}_{k-1} u) \tag{7.13}$$

holds. By Lemma 3.4, $|u - \tilde{Q}_0 u|_1$ and therefore also $|\tilde{Q}_0 u|_1$ can be estimated by $|u|_1$. We have proven the main result of this section:

Theorem 7.5. *Under the assumption* (7.1) *and* (7.2), *there exists a constant* $\tilde{K}_1$, *independent of j, such that*

$$|\tilde{Q}_0 u|_1^2 + \sum_{k=1}^{j} 4^k \|\tilde{Q}_k u - \tilde{Q}_{k-1} u\|_0^2 \leqq \tilde{K}_1 |u|_1^2 \tag{7.14}$$

for all $u \in \mathcal{S}'$.

Of course, the estimates above hold also for the case that no boundary values are prescribed, even if with possibly larger constants.

Together with the results of the preceding two sections, we get the described final lower estimate:

Theorem 7.6. *There exists a constant* K_1 *independent of j such that, for all* $u \in \mathcal{S}$,

$$|Q_0 u|_1^2 + \sum_{k=1}^{j} 4^k \|Q_k u - Q_{k-1} u\|_0^2 \leqq K_1 |u|_1^2. \tag{7.15}$$

Together with Theorem 4.1, this theorem shows that the discrete (semi-)norm

$$|||u|||^2 = |Q_0 u|_1^2 + \sum_{k=1}^{j} 4^k \|Q_k u - Q_{k-1} u\|_0^2 \tag{7.16}$$

on $\mathcal{S}$ is equivalent to the H^1-like seminorm $|u|_1$.

In fact, only the maximum ratio of the coefficients $\omega(T)$ for *neighboring* triangles in the initial triangulation $\mathcal{T}_0$ enters into the optimal constants $\tilde{K}_1$ and K_1 in the last two theorems, and not the quasiuniformity of $\mathcal{T}_0$ or the maximum ratio of the coefficients $\omega(T)$, $T \in \mathcal{T}_0$. This can be seen by a local version of our proof. One has to introduce L_2-bounded quasi-interpolation operators $\tilde{M}_k$: $\tilde{\mathcal{S}} \to \tilde{\mathcal{S}}_k$, which reproduce the functions in $\tilde{\mathcal{S}}_k$, such that for $T \in \tilde{\mathcal{T}}_k$,

$\tilde{M}_k u|T$ depends only on the values of u on the triangles in $\tilde{\mathcal{T}}_k$ intersecting the triangle T. Then one utilizes

$$\|u-\tilde{Q}_k u\|_0^2 \leqq \sum_{T\in\tilde{\mathcal{T}}_k} \|u-\tilde{M}_k u\|_{0;T}^2 \tag{7.17}$$

and estimates the terms in the sum on the right-hand side by local K-functionals. Using uniformly bounded local extension operators, one finally obtains the desired local version of Theorem 7.6.

8. Local decompositions

For the analysis of multigrid methods on nonuniformly refined meshes with local smoothing procedures, and for the analysis of the corresponding additive methods, one needs stable decompositions

$$u=u_0+\sum_{k=1}^{j} u_k, \qquad u_k\in\mathcal{S}_k, \tag{8.1}$$

of the functions $u\in\mathcal{S}$ where the functions u_k vanish at all nodes outside the triangles of a level $\geqq k$ in $\mathcal{T}_j$, i.e. outside a neighborhood of the refinement region. In this section, we construct such decompositions.

The discrete norm on $\mathcal{S}=\mathcal{S}_j$ given by

$$|||v|||_0^2=\frac{1}{12}\sum_{\hat{T}\in\mathcal{T}_0}\omega(\hat{T})\sum_{T\in\mathcal{T}_j,\,T\subseteq\hat{T}}\frac{area(T)}{area(\hat{T})}\sum_{x\in\mathcal{N}_j\cap T}|v(x)|^2 \tag{8.2}$$

is equivalent to the weighted norm (2.3). A simple calculation shows that

$$\tfrac{1}{4}\|v\|_0^2\leqq|||v|||_0^2\leqq\|v\|_0^2, \qquad v\in\mathcal{S}, \tag{8.3}$$

holds. The norm (8.2) induces an inner product on $\mathcal{S}$. With respect to this inner product, we define orthogonal projections $\Pi_k\colon\mathcal{S}\to\mathcal{S}_k$ and consider the decomposition

$$u=\Pi_0 u+\sum_{k=1}^{j}(\Pi_k u-\Pi_{k-1}u) \tag{8.4}$$

of the functions $u\in\mathcal{S}$. $\Pi_k u-\Pi_{k-1}u$ vanishes at all nodes $x\in\mathcal{N}_j$ not contained in triangles of $\mathcal{T}_j$ of a level $\geqq k$. If we are able to show that the seminorm given by

$$|\Pi_0 u|_1^2+\sum_{k=1}^{j}4^k\|\Pi_k u-\Pi_{k-1}u\|_0^2 \tag{8.5}$$

is equivalent to the seminorm (2.11), then (8.4) is a decomposition of the desired type.

Theorem 8.1. *There is a constant K_1^* with*

$$|\Pi_0 u|_1^2 + \sum_{k=1}^{j} 4^k \|\Pi_k u - \Pi_{k-1} u\|_0^2 \leqq K_1^* |u|_1^2 \tag{8.6}$$

and a constant K_2^* with

$$|u|_1^2 \leqq K_2^* \left\{ |\Pi_0 u|_1^2 + \sum_{k=1}^{j} 4^k \|\Pi_{k-1} u - \Pi_k u\|_0^2 \right\}. \tag{8.7}$$

Proof. By (8.3) we have

$$\begin{aligned}\|\Pi_k u - \Pi_{k-1} u\|_0^2 &\leqq 4 |||\Pi_k u - \Pi_{k-1} u|||_0^2 \leqq 4 |||u - \Pi_{k-1} u|||_0^2 \\ &\leqq 4 |||u - Q_{k-1} u|||_0^2 \leqq 4 \|u - Q_{k-1} u\|_0^2.\end{aligned}$$

As in the proof of Theorem 5.2,

$$\sum_{k=1}^{j} 4^k \|\Pi_k u - \Pi_{k-1} u\|_0^2 \leqq \tfrac{16}{3} \sum_{k=1}^{j} 4^k \|Q_k u - Q_{k-1} u\|_0^2$$

follows. Theorem 7.6 yields

$$\sum_{k=1}^{j} 4^k \|\Pi_k u - \Pi_{k-1} u\|_0^2 \leqq c_1 |u|_1^2.$$

Utilizing Lemma 3.4, one obtains as usual

$$|u - \Pi_0 u|_1^2 \leqq c_2 \sum_{k=1}^{j} 4^k \|\Pi_k u - \Pi_{k-1} u\|_0^2.$$

Therefore $|u - \Pi_0 u|_1$ and $|\Pi_0 u|_1$ can also be estimated by $|u|_1$. This proves (8.6). The estimate (8.7) follows from Lemma 3.4. □

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