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Homogenization in Time II: Mechanical Systems Subject to Friction and Gyroscopic Forces

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HOMOGENIZATION IN TIME II: MECHANICAL SYSTEMS SUBJECT TO FRICTION AND GYROSCOPIC FORCES

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ABSTRACT. In our previous work [1] we have studied natural mechanical systems on Riemannian manifolds with a strong constraining potential. These systems establish fast nonlinear oscillations around some equilibrium manifold. Important in applications, the problem of elimination of the fast degrees of freedom, or *homogenization in time*, leads to determine the singular limit of infinite strength of the constraining potential. In the present paper we extend this study to systems which are subject to external forces that are non-potential, depending in a mixed way on positions *and* velocities. We will argue that the method of weak convergence used in [1] covers such forces if and only if they result from viscous friction and gyroscopic terms. All the results of [1] directly extend if there is no friction transversal to the equilibrium manifold; elsewise we show that instructive modifications apply.

INTRODUCTION

Suppose that, due to a strong constraining force, the motion of a mechanical system exhibits rapid oscillations around some equilibrium manifold. These rapid oscillations occur on a time scale τ_{fast} that is small compared to the time scale τ_{avg} of the *average* motion. The large ratio $\epsilon^{-1} = \tau_{\text{avg}}/\tau_{\text{fast}} \gg 1$ then measures the relative strength of the constraining force.

For a variety of reasons, such as a better understanding of the average motion, model simplification and dimensional reduction, or the acceleration of numerical integrators, one is interested in establishing models which approximate the average motion of the mechanical system by a dynamical system *constrained* to the equilibrium manifold; thus *eliminating* the rapidly oscillating degrees of freedom.

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This elimination can be accomplished by determining the singular limit $\epsilon \to 0$ of *infinite* strength of the constraining force. We distinguish two different cases:

- If the fast motion transversal to the equilibrium manifold and the slow motion along that manifold are energetically decoupled for $\epsilon \to 0$, the limit dynamics is just given by the mechanical system that is holonomically constrained to the equilibrium manifold and is subject to all forces of the original system *except* for the strong constraining one. In this case, we say that the strong constraining force *realizes holonomic constraints.*
- If, because of nonlinear coupling and inhomogeneities, there is a slow exchange of energy between the fast transversal motion and the slow motion along the equilibrium manifold, we obtain in addition to the limit description of the previous case a further, nontrivial force term. In this case, we use the term *homogenization in time* for the process of setting up the limit dynamics.

In our work [1] we have studied these two cases in detail for *natural mechanical systems* on Riemannian manifolds, i.e., assuming that all forces are stemming from a potential. We have established that the velocities are in general just weakly convergent, which explains the particular difficulty of the problem of homogenization in time: nonlinear functionals are not weakly sequentially continuous. The defect of this non-continuity causes the additional nontrivial force term, which again turns out to stem from a potential. Moreover, we have completely characterized the initial data and constraining potentials for which there is realization of holonomic constraints.

In the present paper we will generalize all these results to mechanical systems which are subject to additional external forces that are non-potential, depending in a mixed way on velocities and positions. We will argue that the method of [1] extends if and only if these forces result from viscous friction and gyroscopic terms. If there is no friction transversal to the equilibrium manifold, all the results of [1] directly extend; whereas elsewise instructive modifications of the result of homogenization in time apply.

The paper is organized as follows: In Sec. 1, we shortly review the notions and results of [1, Chap. II]. In Sec. 2, we study the admissible form of the external force terms to which the method applies. In Sec. 3, we state and prove the generalization of homogenization in time. Finally, in Sec. 4, we discuss the generalization of realization of holonomic constraints.

1. A Review of the Results for Natural Mechanical Systems

Let M be a smooth Riemannian manifold with metric $\langle\cdot,\cdot\rangle$. For a sequence $\epsilon \to 0$, we consider a family of natural mechanical systems on the configuration space M given by the Lagrangians

$$\mathscr{L}_\epsilon(x,\dot{x}) = \tfrac{1}{2}\langle\dot{x},\dot{x}\rangle - V(x) - \epsilon^{-2}U(x), \qquad \dot{x} \in T_xM,$$

with smooth potentials V and U. We assume that the potential V is bounded from below and the non-negative potential U possesses a nondegenerate critical submanifold $N \subset M$.[1] In this case we call U *constraining to* N. This further qualifies to the term *constraining spectrally smooth to* N if the Hessian H of U has a smooth spectral decomposition on N,[2]

$$H(x) = \sum_{\lambda=1}^{s} \omega_\lambda^2(x) P_\lambda(x), \qquad x \in N.$$

Because N is *nondegenerate*, we get that

$$P(x) = \sum_{\lambda=1}^{s} P_\lambda(x) : T_x M \to T_x N^\perp, \qquad x \in N,$$

is the orthogonal projection of $T_x M$ *onto* $T_x N^\perp$ and that there is a constant $\omega_* > 0$ such that the smooth eigen-frequencies ω_λ are bounded from below as $\omega_\lambda(x) \geq \omega_*$ $(x \in N)$.

Corresponding to the family of Lagrangians, the singularly perturbed equations of motion are given by the Euler-Lagrange equations

$$\nabla_{\dot{x}_\epsilon} \dot{x}_\epsilon + \operatorname{grad} V(x_\epsilon) + \epsilon^{-2} \operatorname{grad} U(x_\epsilon) = 0, \tag{1}$$

where the *covariant derivative* ∇ denotes the Levi-Cività connection of the Riemannian manifold M. The *energy*

$$E_\epsilon = \tfrac{1}{2}\langle \dot{x}_\epsilon, \dot{x}_\epsilon \rangle + V(x_\epsilon) + \epsilon^{-2} U(x_\epsilon),$$

is an invariant of motion. For physical reasons, we bound the energy uniformly in ϵ, $E_\epsilon \leq E_*$. This is, in fact, a condition on the initial values, which for simplicity we choose to be fixed in the positions and converging in the velocities,

$$x_\epsilon(0) = x_*, \qquad \lim_{\epsilon \to 0} \dot{x}_\epsilon(0) = v_* \in T_{x_*} M. \tag{2}$$

The equi-boundedness of the energy directly implies that $U(x_*) = 0$, i.e., $x_* \in N$. Upon introducing the constants

$$\theta_*^\lambda = \frac{\langle P_\lambda(x_*) v_*, P_\lambda(x_*) v_* \rangle}{2\,\omega_\lambda(x_*)}, \qquad \lambda = 1, \ldots, s,$$

we define the *homogenization* of the constraining potential U with respect to the initial values (2) by

$$U_{\text{hom}}(x) = \sum_{\lambda=1}^{s} \theta_*^\lambda \, \omega_\lambda(x), \qquad x \in N. \tag{3}$$

[1] I.e., the smoothly embedded submanifold $N \subset M$ satisfies $N = \{x \in M : U(x) = 0\} = \{x \in M : \operatorname{grad} U(x) = 0\}$ and the Hessian H of U, defined as a field of linear operators $H : TM|N \to TM|N$ by $\langle H(x)u, v \rangle = D^2 U(x)(u, v)$ $(u, v \in T_x M,\ x \in N)$ fulfills the nondegeneracy condition $\ker H(x) = T_x N$ $(x \in N)$.

[2] The smooth bundle maps $P_\lambda : TM|N \to TN^\perp$ define by $P_\lambda(x) : T_x M \to T_x N^\perp$ $(x \in N)$ orthogonal projections of $T_x M$ onto mutually orthogonal subspaces of $T_x N^\perp$.

Now, the *homogenization in time* of the sequence of mechanical systems given by $\mathscr{L}_\epsilon$ with respect to the initial values (2) is the mechanical system corresponding to the Lagrangian

$$\mathscr{L}_{\text{hom}}(x, \dot{x}) = \tfrac{1}{2}\langle \dot{x}, \dot{x}\rangle - V(x) - U_{\text{hom}}(x), \qquad \dot{x} \in T_x N.$$

We denote by x_{hom} the solution of the corresponding Euler-Lagrange equations with initial values $x_{\text{hom}}(0) = x_*$, $\dot{x}_{\text{hom}}(0) = v_* - P(x_*)v_*$.

On the other hand, we define as x_{con} the solution of the Euler-Lagrange equations that belong to the Lagrangian

$$\mathscr{L}_{\text{con}}(x, \dot{x}) = \tfrac{1}{2}\langle \dot{x}, \dot{x}\rangle - V(x), \qquad \dot{x} \in T_x N,$$

subject to the same initial values as x_{hom}. We speak of *realization of holonomic constraints* if $x_\epsilon \to x_{\text{con}}$, uniformly on the time interval under consideration.

The main results of [1, Chap. II], namely Theorems II.1–3, can be summarized as follows:

Theorem 1. *Consider a finite time interval* $[0, T]$.

(i) *Let U constrain spectrally smooth to N. If x_{hom} is non-flatly resonant up to order three,*[3] *the sequence x_ϵ converges uniformly to x_{hom}.*

(ii.a) *Let the initial velocity $\dot{x}_\epsilon(0) = v_* \in T_{x_*}M$ be independent of ϵ. If and only if $v_* \in T_{x_*}N$, there is realization of holonomic constraints for all potentials U that constrain to N.*

(ii.b) *Let U constrain to N. Suppose the spectrum $\sigma(H)$ of the Hessian is constant on N. Then, the potential U constrains spectrally smooth and, for all fixed initial velocities $\dot{x}_\epsilon(0) = v_* \in T_{x_*}M$, realizes holonomic constraints.*

Note that, in general, the velocities converge just weakly* in L^∞ with respect to coordinates of a local bundle trivialization of TM. In particular, [1, Lemma II.17] shows that, for fixed initial velocity $\dot{x}_\epsilon(0) = v_* \in T_{x_*}M$, this weak* convergence is strong *if and only if* we have $v_* \in T_{x_*}N$. This situation is exactly the case (ii.a) of the theorem above.

2. The Admissible Form of External Forces

In the present paper we allow for force fields that are more general than potential ones. Therefore, we consider the dynamics of a mechanical system that is governed by the Lagrangian

$$\mathscr{L}_\epsilon(x, \dot{x}) = \tfrac{1}{2}\langle \dot{x}, \dot{x}\rangle - V(x) - \epsilon^{-2}U(x), \qquad \dot{x} \in T_x M,$$

[3] I.e., if there is a time $t_r \in [0, T]$ such that $\gamma_1\omega_1(x_{\text{hom}}(t_r)) + \ldots + \gamma_s\omega_s(x_{\text{hom}}(t_r)) = 0$ for integers $\gamma_\lambda \in \mathbb{Z}$ with $1 \le |\gamma_1| + \ldots + |\gamma_s| \le 3$, there holds the non-flatness condition

$$\frac{d}{dt}\left(\gamma_1\omega_1(x_{\text{hom}}) + \ldots + \gamma_s\omega_s(x_{\text{hom}})\right)\Big|_{t=t_r} \neq 0.$$

subject to an additional *external force* of the form

$$F(x, \dot{x}) \in T_x M, \qquad \dot{x} \in T_x M.$$

Instead of (1), the equations of motion are by definition given as

$$\nabla_{\dot{x}_\epsilon} \dot{x}_\epsilon + \operatorname{grad} V(x_\epsilon) + \epsilon^{-2} \operatorname{grad} U(x_\epsilon) = F(x_\epsilon, \dot{x}_\epsilon). \tag{4}$$

However, we cannot attack the problem of homogenization in quite that generality.

If the energy arguments of [1, §II.2.1] apply, we will get the convergences

$$x_\epsilon \to x_0, \qquad \dot{x}_\epsilon \stackrel{*}{\rightharpoonup} \dot{x}_0.$$

Thus, the force F, depending on the just weakly* converging velocities, will have a nontrivial impact on the limit dynamics. For our method to work this impact should only appear on the level of energies, but not on the level of forces. This means, for obtaining an abstract limit equation analogous to the one given in [1, Lemma II.8], we have to require the weak* continuity

$$F(x_\epsilon, \dot{x}_\epsilon) \stackrel{*}{\rightharpoonup} F(x_0, \dot{x}_0).$$

The most general force that guarantees this weak* continuity is *affine* in the velocities, [2, Theorem. I.1.1]:

$$-F(x, \dot{x}) = F_0(x) + K(x) \cdot \dot{x},$$

where $K : TM \to TM$ is a field of linear operators.

The total energy of the system (4), again an *invariant* of motion, is given by the expression

$$E_\epsilon = \tfrac{1}{2}\langle \dot{x}_\epsilon, \dot{x}_\epsilon \rangle + V(x_\epsilon) + \epsilon^{-2} U(x_\epsilon) - \int_0^t \langle F(x_\epsilon, \dot{x}_\epsilon), \dot{x}_\epsilon \rangle \, d\tau. \tag{5}$$

For the energy arguments of [1, §II.2.1] to apply, we have to infer from the boundedness $E_\epsilon \le E_*$ a corresponding bound of just the kinetic energy $\frac{1}{2}\langle \dot{x}_\epsilon, \dot{x}_\epsilon \rangle$. Therefore, the last term in (5) has to be bounded from below, independently of the specification of x_ϵ. This way we obtain further restrictions of the force term F. Quite the most general admissible form is given as follows:[4]

- F_0 is a potential force, belonging to a potential that is bounded from below,

[4] To be specific: For all $x_* \in M$, $v_* \in T_{x_*}M$ and times T there has to be a constant $\beta = \beta(x_*, v_*, T)$ such that

$$-\int_0^t \langle F(x(\tau), \dot{x}(\tau)), \dot{x}(\tau) \rangle \, d\tau \ \ge\ \beta, \qquad 0 \le t \le T,$$

where $x : [0, T] \to M$ is any given smooth path starting as $x(0) = x_*$, $\dot{x}(0) = v_*$. This readily implies the nonnegativity of K. Moreover, if K is skew-adjoint with respect to the Riemannian metric, F_0 has *necessarily* to be a potential force, belonging to a potential that is bounded from below.

- the linear operator K is *nonnegative*, i.e., $\langle K(x)v, v\rangle \geq 0$ for all tangential vectors $v \in T_xM$.

We may put $F_0 = 0$ by incorporating the corresponding potential into the weak potential V of the Lagrangian $\mathscr{L}_\epsilon$.

We split the nonnegative operator $K = A + S$ into its nonnegative selfadjoint part A and its skew-adjoint part S,

$$A = \tfrac{1}{2}(K + K^*) \geq 0, \qquad S = \frac{1}{2}(K - K^*).$$

Correspondingly, there is the splitting

$$F(x, \dot{x}) = F_{\text{fric}}(x, \dot{x}) + F_{\text{gyro}}(x, \dot{x})$$

of the force $F(x, \dot{x}) = -K(x)\dot{x}$ into *viscous friction* $F_{\text{fric}}(x, \dot{x}) = -A(x)\dot{x}$, defined by $A = A^* \geq 0$, and a *gyroscopic force* $F_{\text{gyro}}(x, \dot{x}) = -S(x)\dot{x}$, defined by $S = -S^*$. This way, we take two major classes of velocity-dependent forces into account that are of importance in applications.

3. Homogenization in Time

For the sake of simplicity of the result, we restrict ourselves to a specific class of viscous friction.

Definition. A force field $F_{\text{fric}}(x, \dot{x}) = -A(x)\dot{x}$ ($A = A^* \geq 0$) of viscous friction is called *κ-isotropic* transversal to the submanifold $N \subset M$ if there is some real constant $\kappa \geq 0$ such that $\langle A(x)v, v\rangle = \kappa\langle v, v\rangle$ for all $x \in N$ and $v \in T_xN^\perp$.

Using the projection P, we can rewrite the κ-isotropy as

$$P(x)A(x)P(x) = \kappa \cdot P(x), \qquad x \in N. \tag{6}$$

This time we define the *homogenization in time* of the mechanical system given by $\mathscr{L}_\epsilon$ subject to the external force $F = F_{\text{fric}} + F_{\text{gyro}}$ as the mechanical system that corresponds to the *non-autonomous* Lagrangian

$$\mathscr{L}^\kappa_{\text{hom}}(x, \dot{x}, t) = \tfrac{1}{2}\langle \dot{x}, \dot{x}\rangle - V(x) - e^{-\kappa t}U_{\text{hom}}(x), \qquad \dot{x} \in T_xN,$$

subject to the external force $F^N = (I - P)F$. This projected force is, in fact, only evaluated on TN which means that the nature of the parts F_{fric} and F_{gyro} does not change: Having $\dot{x} \in T_xN$, the projected part

$$F^N_{\text{fric}}(x, \dot{x}) = -A^N(x)\dot{x}, \qquad A^N = (I - P)A(I - P),$$

constitutes viscous friction on the configuration space N, whereas

$$F^N_{\text{gyro}}(x, \dot{x}) = -S^N(x)\dot{x}, \qquad S^N = (I - P)S(I - P),$$

defines a gyroscopic force term on the configuration space N. We denote by x_{hom} the solution of the corresponding equations of motion with initial values $x_{\text{hom}}(0) = x_*$, $\dot{x}_{\text{hom}}(0) = v_* - P(x_*)v_*$.

Now, part (i) of Theorem 1 generalizes to systems with κ-isotropic friction and gyroscopic forces as follows.

Theorem 2. *Consider a finite time interval* $[0, T]$. *Suppose* U *constrains spectrally smooth to* N *and the force field* F_{fric} *of viscous friction is* κ-*isotropic transversal to* N. *If* x_{hom} *is non-flatly resonant up to order three, the sequence* x_ϵ *converges uniformly to* x_{hom}.

We can interpret the assertion of this theorem depending on which value the constant κ of friction takes:

- $\kappa = 0$, the "purely gyroscopic" case transversal to N. Here, we have $\mathscr{L}^{\kappa}_{\text{hom}} = \mathscr{L}_{\text{hom}}$ which means that [1, Theorem II.1] generalizes in perfect analogy.
- $\kappa > 0$, the case of friction transversal to N. Here, the presence of the homogenized potential *fades out*, exponentially fast in time; yielding a continuous transition to the Lagrangian of holonomic constraints,

  $$\mathscr{L}^{\kappa}_{\text{hom}}|_{t=0} = \mathscr{L}_{\text{hom}}, \qquad \lim_{t\to\infty} \mathscr{L}^{\kappa}_{\text{hom}} = \mathscr{L}_{\text{con}}.$$

  This way, we have given a precise mathematical meaning to the physical argument of Koppe and Jensen [3, p. 8] that any strong potential *finally* realizes holonomic constraints if there is friction in the transversal motion.[5]

Proof of Theorem 2. The proof may be obtained by modifying the proof of [1, Theorem II.1] appropriately. For that proof of 21 pages is far too long to be restated here, we will just highlight these modifications; all the other results of [1, §II.2] remain valid with literally the same proof.

Therefore, even a reader who just wants to understand the notation used here would be advised to open the monograph [1] at page 29 and read the following account on the four steps of proof in parallel.

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[5] Koppe and Jensen write (loc. cit.): "Da aber die ganze Technische Mechanik durch das d'Alembertsche Prinzip beherrscht wird und es wohl nur wenige besser empirisch bestätigte Naturgesetze gibt, ist notwenigerweise außer der Annahme eines Führungspotentials (bzw. von Führungskräften) zu seiner Begründung noch ein weiterer Gesichtspunkt erforderlich. Dieser scheint uns darin zu liegen, daß die [...] ‚Energie der Transversalbewegung' wegen der hohen Frequenz, mit der in ihr sich kinetische und potentielle Energie ineinander umsetzen, sehr rasch dissipiert wird, auch wenn wir von der Dämpfung der Bewegung längs der Führungsgeraden noch ganz absehen können. Infolgedessen wird, wie auch immer die Bewegung gestartet sein mag, [die transversale Energie] durch Dämpfung rasch gegen Null gehen, und dann die weitere Bewegung so verlaufen, wie sie durch das d'Alembertsche Prinzip bestimmt ist."

[Translation by the author: "Because all of the mechanics in technical applications is governed by d'Alembert's principle, and hardly a law of nature is empirically better confirmed, there is necessarily a further aspect required in addition to the assumption of a strong constraining potential (resp. a strong constraining force). In our view, this aspect seemingly lies in the fact that the 'energy of transversal motion' is rapidly dissipated—even if we may completely neglect all damping of the motion along the constraint manifold—because kinetic energy is converted to potential energy and vice versa with high frequency. Therefore, however the motion is started, the transversal energy will rapidly be damped to zero and then, afterwards, the motion will take place according to d'Alembert's principle."]

Step 1: Equi-Boundedness (as in [1, §II.2.1]*).* The energy (5) of the system specifies as

$$E_\epsilon = \tfrac{1}{2}\langle \dot{x}_\epsilon, \dot{x}_\epsilon\rangle + V(x_\epsilon) + \epsilon^{-2}U(x_\epsilon) + \int_0^t \langle K(x_\epsilon)\dot{x}_\epsilon, \dot{x}_\epsilon\rangle\, d\tau.$$

Because $K(x)$ is a nonnegative linear operator, the integral is always nonnegative.

Step 2: The Weak Virial Theorem (as in [1, §II.2.2]*).* First, we note that the splitting of energy, as given in [1, Lemma II.6], has to be defined differently. However, this modification leaves the definition of the energies of normal (transversal) motion, $E_\epsilon^\perp = T_\epsilon^\perp + U_\epsilon^\perp$, *untouched.* Since [1, Lemma II.6] has no impact on the proof other than justifying the *notion* of energy components, we postpone the detailed discussion of the modified energy splitting to the discussion of realization of holonomic constraints where it will be needed.

Second, the abstract limit equation [1, (II.30)] of [1, Lemma II.8] changes to

$$\ddot{x}_0 + \Gamma(x_0)(\dot{x}_0, \dot{x}_0) + F_V(x_0) + F_U^{\text{hom}}(t) - F(x_0, \dot{x}_0) \ \perp\ T_{x_0}N. \tag{7}$$

Here, we have made use of $F(x, \dot{x}) = -K(x)\dot{x}$ being *linear* in the velocity argument.

Step 3: Adiabatic "Invariance" of the Normal Actions (as in [1, §II.2.3]*).* The detailed resolution of the right-hand-side of the componentwise oscillator equations [1, (II.45)], as given by [1, Lemma II.12], *is* affected by the velocity-dependence of the external force term $F(x, \dot{x})$. Because of

$$P_{\epsilon\lambda}F(x_\epsilon, \dot{x}_\epsilon) \equiv -P_{\epsilon\lambda}K_\epsilon \dot{z}_\epsilon \quad (\text{mod } C^0\text{-lim})$$

there appears an additional term, leading to the modified equation

$$\begin{aligned} \ddot{z}^i_{\epsilon\lambda} + \epsilon^{-2}\omega^2_{\epsilon\lambda} z^i_{\epsilon\lambda} &= -P_{\epsilon\lambda}K_\epsilon \dot{z}_\epsilon + 2(\dot{P}_{\epsilon\lambda}\dot{z}_\epsilon)^i - 2(P_{\epsilon\lambda}\Gamma_\epsilon(\dot{y}_\epsilon, \dot{z}_\epsilon))^i \\ &\quad + a_{\epsilon\lambda}{}^i_{jk}\dot{z}^j_\epsilon \dot{z}^k_\epsilon + \epsilon^{-2} b_{\epsilon\lambda}{}^i_{jk} z^j_\epsilon z^k_\epsilon + c_{\epsilon\lambda}{}^i. \end{aligned} \tag{8}$$

This additional term $-P_{\epsilon\lambda}K_\epsilon\dot{z}_\epsilon$ is the reason, that there is no adiabatic *invariance* in [1, Lemma II.14] any more. Instead, there holds the following exponential *fade-out* of the normal actions

$$\theta^\lambda_\epsilon \ \to\ \theta^\lambda_0 = e^{-\kappa t}\theta^\lambda_*.$$

The rest of Step 3 provides a proof for this assertion. If we follow the proof of [1, Lemma II.14] with the evaluation of $\dot{E}^\perp_{\epsilon\lambda}$, there appears the additional term

$$\begin{aligned} s_a = -\operatorname{tr}(P_{\epsilon\lambda}K_\epsilon \dot{z}_\epsilon \otimes \dot{z}_{\epsilon\lambda}G_\epsilon) &= -\sum_\mu \operatorname{tr}(P_{\epsilon\lambda}K_\epsilon \dot{z}_{\epsilon\mu} \otimes \dot{z}_{\epsilon\lambda}G_\epsilon) \\ &= -\sum_\mu \operatorname{tr}(K_\epsilon P_{\epsilon\mu}\Pi_\epsilon P_{\epsilon\lambda}) + O(\epsilon). \end{aligned}$$

By [1, Eq. (II.37)], we have the weak*-limit $s_a \overset{*}{\rightharpoonup} -\operatorname{tr}(K_0 P_{0\lambda}\Pi_0 P_{0\lambda})$. Now, the splitting $K = A + S$, with S being skew, yields

$$\operatorname{tr}(K_0 P_{0\lambda}\Pi_0 P_{0\lambda}) = \operatorname{tr}(A_0 P_{0\lambda}\Pi_0 P_{0\lambda}) = \kappa \operatorname{tr}(P_{0\lambda}\Pi_0 P_{0\lambda}) = \kappa\sigma_\lambda\omega_{0\lambda}^2.$$

Here, we have used the assumption that A is κ-isotropic, which allows to simplify $P_{0\lambda}A_0 P_{0\lambda} = \kappa\cdot P_{0\lambda}$ by (6). Summarizing, we obtain the weak*-limit

$$\dot{E}_{\epsilon\lambda}^{\perp} \overset{*}{\rightharpoonup} \dot{E}_{0\lambda}^{\perp} = -\kappa\sigma_\lambda\omega_{0\lambda}^2 + \tfrac{1}{2}\sigma_\lambda\frac{d}{dt}\omega_{0\lambda}^2.$$

A comparison with a direct differentiation of the expression $E_{0\lambda}^{\perp} = \sigma_\lambda\omega_{0\lambda}^2$, that is

$$\dot{E}_{0\lambda}^{\perp} = \dot{\sigma}_\lambda\omega_{0\lambda}^2 + \sigma_\lambda\frac{d}{dt}\omega_{0\lambda}^2,$$

yields the following equation of logarithmic differentials:

$$\frac{\dot{\sigma}_\lambda}{\sigma_\lambda} = -\kappa - \frac{\dot{\omega}_{0\lambda}}{\omega_{0\lambda}}.$$

Thus, there are constants θ_*^λ such that

$$\sigma_\lambda = \frac{e^{-\kappa t}\theta_*^\lambda}{\omega_{0\lambda}}, \qquad E_{0\lambda}^{\perp} = e^{-\kappa t}\theta_*^\lambda\omega_{0\lambda}, \qquad \theta_0^\lambda = e^{-\kappa t}\theta_*^\lambda.$$

As in the proof of [1, Lemma II.14], these constants can be calculated from the initial values by evaluating the limit of the energies $E_{\epsilon\lambda}^{\perp}$ at the initial time $t = 0$. To be specific, we obtain

$$E_{\epsilon\lambda}^{\perp}(0) = T_{\epsilon\lambda}^{\perp}(0) \;\to\; \tfrac{1}{2}\langle P_\lambda(x_*)v_*, P_\lambda(x_*)v_*\rangle,$$

which yields the values

$$\theta_*^\lambda = \theta_0^\lambda(0) = \lim_{\epsilon\to 0}\frac{E_{\epsilon\lambda}^{\perp}(0)}{\omega_\lambda(y_\epsilon(0))} = \frac{\langle P_\lambda(x_*)v_*, P_\lambda(x_*)v_*\rangle}{2\,\omega_\lambda(x_*)},$$

just as in [1, Definition II.4].

Step 4: Identification of the Limit Mechanical System (as in [1, §II.2.4]*).* The assertion of [1, Lemma II.15] has to be modified as

$$Q(x_0)F_U^{\text{hom}} = e^{-\kappa t}\operatorname{grad}_N U_{\text{hom}}(x_0).$$

This can be proven as follows: literally as in the proof of [1, Lemma II.15], we first obtain

$$\tfrac{1}{2}\operatorname{tr}\left(\frac{\partial H(x_0)}{\partial y^i}\cdot\Sigma_0\right) = \sum_\lambda \sigma_\lambda\omega_\lambda(x_0)\cdot\frac{\partial\omega_\lambda(x_0)}{\partial y^i}.$$

Next, inserting the relation $\sigma_\lambda = e^{-\kappa t}\theta_*^\lambda/\omega_\lambda(x_0)$ of the previous step yields the desired result. □

4. Realization of Holonomic Constraints

Here, we define x_{con} as the solution of the equations of motion that belong to the Lagrangian

$$\mathscr{L}_{\text{con}}(x, \dot{x}) = \tfrac{1}{2}\langle \dot{x}, \dot{x}\rangle - V(x), \qquad \dot{x} \in T_x N,$$

subject to the projected external force F^N and the same initial values as x_{hom}. We will do so for all external forces of the form $F(x, \dot{x}) = -K(x)\dot{x}$, K being nonnegative; i.e., we drop the assumption of κ-isotropy of the friction from now on. Again, we speak of *realization of holonomic constraints* if $x_\epsilon \to x_{\text{con}}$, uniformly on the time interval under consideration.

Now, parts (ii.a) and (ii.b) of Theorem 1 generalize as follows.

Theorem 3. *Consider a finite time interval* $[0, T]$.

(a) *Let the initial velocity* $\dot{x}_\epsilon(0) = v_* \in T_{x_*}M$ *be independent of* ϵ. *If and only if* $v_* \in T_{x_*}N$, *there is realization of holonomic constraints for all potentials* U *that constrain to* N *and all external forces* $F(x, \dot{x}) = -K(x)\dot{x}$ *(*K *nonnegative).*

(b) *Let* U *constrain to* N. *Suppose the spectrum* $\sigma(H)$ *of the Hessian is constant on* N. *Then, the potential* U *constrains spectrally smooth and, for all external forces* $F(x, \dot{x}) = -K(x)\dot{x}$ *(*K *nonnegative) and all fixed initial velocities* $\dot{x}_\epsilon(0) = v_* \in T_{x_*}M$, *realizes holonomic constraints.*

Note that no resonance condition applies.

Finally again, we remark that case (a) completely characterizes the conditions for strong convergence of the velocities. For [1, Lemma II.17] remains true, with literally the same proof.

Proof of Theorem 3. As in the proof of Theorem 2 we only sketch the necessary modifications of the proofs of the corresponding Theorems II.2 and II.3 in [1]. An interested reader would be advised to read [1, §§II.3.1–2] in parallel.

It is important to notice that we have not used κ-isotropy during the first two steps of the proof of Theorem 2.

As promised in Step 2 of the proof of Theorem 2, we start by shortly discussing the splitting of energies that modifies [1, Lemma II.6].

Leaving the notion of the energy $E_\epsilon^\perp = T_\epsilon^\perp + U_\epsilon^\perp$ of the normal motion as in [1, Definition II.5], we modify the energy of the constrained motion to

$$E_\epsilon^{\|} = \tfrac{1}{2}\langle \dot{y}_\epsilon, \dot{y}_\epsilon\rangle + V(y_\epsilon) + \int_0^t \langle K(y_\epsilon)\dot{y}_\epsilon, \dot{y}_\epsilon\rangle \, d\tau$$

and additionally define the energy that is *dissipated* in normal direction,

$$E_\epsilon^{\text{diss}} = \int_0^t \operatorname{tr}(K_\epsilon(\tau) \cdot \Pi_\epsilon(\tau)) \, d\tau \;\to\; E_0^{\text{diss}} = \int_0^t \operatorname{tr}(K(x_0) \cdot \Pi_0) \, d\tau.$$

By slightly adjusting the proof of [1, Lemma II.6], one obtains the following result: The total energy E_ϵ decomposes into $E_\epsilon = E_\epsilon^{\|} + E_\epsilon^{\perp} + E_\epsilon^{\mathrm{diss}} + o(1)$ as $\epsilon \to 0$. All three components converge uniformly as functions in $C[0,T]$.

Now, using the abstract limit equation (7) and arguing analogously to the proof of [1, Eq. (II.48)], we obtain

$$
\begin{aligned}
E_0^{\perp}(t) &= -E_0^{\mathrm{diss}}(t) + E_0^{\perp}(0) + \tfrac{1}{2}\int_0^t \langle \dot{x}_0(\tau), \operatorname{grad} H(x_0(\tau)) : \Sigma_0(\tau)\rangle \, d\tau \\
&\le E_0^{\perp}(0) + \tfrac{1}{2}\int_0^t \langle \dot{x}_0(\tau), \operatorname{grad} H(x_0(\tau)) : \Sigma_0(\tau)\rangle \, d\tau.
\end{aligned} \tag{9}
$$

For the last estimate, we notice that the nonnegativity of K implies that $E_\epsilon^{\mathrm{diss}} \ge 0$ which in turn yields $E_0^{\mathrm{diss}} \ge 0$.

This estimate allows to prove part (a) completely analogously to [1, Theorem II.2]. We just have to use estimate (9) instead of [1, Eq. (II.48)].

Likewise, part (b) is proved in complete analogy to [1, Theorem II.3] by using the abstract limit equation (7) instead of [1, Eq. (II.30)]. □

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