A NOTE ON MICROLOCAL OBSTRUCTIONS TO THE COMPACTNESS OF SEQUENCES

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Abstract. This note elaborates some of the remarks made by Patrick Gérard [Gér89, Gér91] about a microlocal understanding of obstructions to the relative compactness of sequences in $L^2_{\rm loc}(\Omega)$. In particular we show how to define the notions of wave front sets and polarization sets of compactness in analogy to the microlocal regularity theory of distributions. The general compensated compactness result of Gérard can be stated in a form that is based on an assumption about the polarization set of the underlying sequence of vector-valued functions. This adds further understanding of the nature of the set Λ in Luc Tartar's Heriot-Watt lecture notes [Tar79] about compensated compactness.

1. Microlocalization of the Kolmogorov Compactness Criterion

The Kolmogorov criterion states that a bounded sequence $u_n \in L^2_{\rm loc}(\Omega)$ is relatively compact iff

$$\forall \phi \in C_0^\infty(\Omega): \qquad \sup_n \|(\phi u_n)(\cdot + h) - \phi u_n\|_{L^2} \to 0 \text{ for } h \to 0.$$

Fourier transform yields the equivalent criterion

$$\forall \phi \in C_0^\infty(\Omega): \qquad \sup_n \int_{|\xi|\geq R} |\widehat{\phi u_n}(\xi)|^2 \, d\xi \to 0 \text{ for } R \to \infty.$$

This suggests the following microlocalization of the criterion.

Definition 1. $\mathrm{WF}(u_n) \subset T^*\Omega \backslash 0$ *is the wave front set of obstructions to microlocal compactness. The sequence u_n is microlocally compact in $L^2_{\rm loc}(\Omega)$ at (x_0, ξ_0), in notation $(x_0, \xi_0) \notin \mathrm{WF}(u_n)$, iff there is a $\phi \in C_0^\infty(\Omega)$ with $\phi(x_0) \neq 0$ and a conical neighborhood Γ of ξ_0 such that*

$$\sup_n \int_{\xi\in\Gamma:|\xi|\geq R} |\widehat{\phi u_n}(\xi)|^2 \, d\xi \to 0 \textit{ for } R \to \infty.$$

The notion $\mathrm{WF}(u_n)$ is independent of the choice of ϕ and Γ because of the following Lemma.

Lemma 1. *There is*

$$\mathrm{WF}(u_n) = \bigcap \mathrm{char}\, A,$$

where the intersection runs over all pseudodifferential operators $A \in \Psi^0$ of order zero such that Au_n is relatively compact in $L^2_{\rm loc}(\Omega)$.

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Proof. We base our proof on analogous arguments in the theory of microlocal regularity of distributions [GS94, Prop. 7.8, p. 78].
(a) Let $(x_0, \xi_0) \notin \mathrm{WF}(u_n)$. Choose ϕ and Γ as in the definition of WF. Let $\chi \in S^0(\mathbb{R}^n)$ be a function supported in Γ that is homogeneous for large ξ and fulfills

$$0 \le \chi \le 1, \qquad \chi(t\xi_0) = 1 \text{ for large } |t|.$$

By construction, we have $A(x, D) = \chi(D)\phi(x) \in \Psi^0$ with

$$(x_0, \xi_0) \notin \mathrm{char}\, A.$$

Now, the estimate

$$\begin{aligned}\sup_n \int_{|\xi|\ge R} |\widehat{A(x,D)u_n}(\xi)|^2\, d\xi &= \sup_n \int_{|\xi|\ge R} |\chi(\xi)|^2 |\widehat{\phi u_n}(\xi)|^2\, d\xi \\ &\le \sup_n \int_{\xi\in\Gamma:|\xi|\ge R} |\widehat{\phi u_n}(\xi)|^2\, d\xi \to 0 \text{ for } R \to \infty\end{aligned}$$

shows that Au_n is relatively compact. Thus, $\complement\mathrm{WF}(u_n) \subset \bigcup \complement\mathrm{char}\, A$ follows.
(b) Let now be $A \in \Psi^0$ such that Au_n is relatively compact and $(x_0, \xi_0) \notin \mathrm{char}\, A$. For $\phi \in C_0^\infty(\Omega)$, $\chi \in S^0$ as above but with sufficiently small support and $\chi \equiv 1$ on a conical neighborhood Γ of ξ_0, we can find a microlocal parametrix B such that

$$\chi(D)\phi = B \circ A + R, \qquad R \in \Psi^{-\infty}.$$

Because R is compact, $\chi(D)\phi u_n$ is relatively compact and we can estimate

$$\begin{aligned}\sup_n \int_{\xi\in\Gamma:|\xi|\ge R} |\widehat{\phi u_n}(\xi)|^2\, d\xi &\le \sup_n \int_{|\xi|\ge R} |\chi(\xi)|^2 |\widehat{\phi u_n}(\xi)|^2\, d\xi \\ &= \sup_n \int_{|\xi|\ge R} |\widehat{\chi(D)\phi u_n}(\xi)|^2\, d\xi \to 0 \text{ for } R \to \infty,\end{aligned}$$

which proves $\complement\mathrm{WF}(u_n) \supset \bigcup \complement\mathrm{char}\, A$. □

A few consequences:

A. Define $\mathrm{defsupp}\, u_n$ as the smallest closed set, where the sequence u_n is not relatively compact. The *same* proof [GS94, Prop. 7.3, p. 78] as for the singular support of distributions shows that

$$\mathrm{defsupp}\, u_n = \pi(\mathrm{WF}(u_n)).$$

B. For $A \in \Psi^0$ define $\mathrm{WF}_{-1}(A)$ as the smallest closed cone $\Gamma \subset T^*\Omega \setminus 0$, such that $\sigma_A|_{\complement\Gamma} \in S^{-1}(\complement\Gamma)$. The *same* proof [GS94, Lemma 7.2, p. 77] as for microlocal regularity shows that

$$\forall A \in \Psi^0: \qquad \mathrm{WF}(Au_n) \subset \mathrm{WF}_{-1}(A) \cap \mathrm{WF}(u_n). \tag{1}$$

C. There holds

$$\forall A \in \Psi^0: \qquad \mathrm{WF}(u_n) \subset \mathrm{WF}(Au_n) \cup \mathrm{char}\, A. \tag{2}$$

Proof. If $(x_0, \xi_0) \notin \mathrm{WF}(Au_n)$ there is a $B \in \Psi^0$ with $(x_0, \xi_0) \notin \mathrm{char}\, B$ and BAu_n relatively compact. If already $(x_0, \xi_0) \notin \mathrm{char}\, A$, it follows that $(x_0, \xi_0) \notin \mathrm{char}\, BA$, i.e., by Lemma 1 $(x_0, \xi_0) \notin \mathrm{WF}(u_n)$. □

D. If u_n solves the hyperbolic Cauchy problem

$$\partial_t u_n + i\,a(t,x,D)u_n = 0, \qquad u(0,\cdot) = \phi_n,$$

where $a \in S^1_{\text{phg}}$ is polyhomogeneous of first order with real principal symbol. The *same* proof [Hör85, p. 388ff] as for microlocal regularity shows that

$$\text{WF}(u_n(t,\cdot)) = \chi_t \text{WF}(\phi_n),$$

where χ_t denotes the bicharacteristic flow belonging to the principal symbol. In particular, this implies that in the high-frequency limit oscillations of the initial values are transported along the bicharacteristic flow, a result that was proven for the wave-equation using much more elaborate techniques in [FM92].

E. Since elliptic Fourier integral operators F of zero order are locally a propagator of a hyperbolic Cauchy problem, we obtain

$$\text{WF}(Fu_n) = \chi_F \text{WF}(u_n),$$

where χ_F denotes the canonical transformation belonging to F. For an elliptic $A \in \Psi^0$ we thus obtain

$$\text{WF}(Au_n) = \text{WF}(u_n),$$

which can alternatively be proven using (1), (2), and the fact that for elliptic pseudodifferential operators we have $\text{char}\, A = \emptyset$.

2. Microlocal Defect Measure

Let $u_n \in L^2_{\text{loc}}(\Omega, \mathbb{C}^N)$ be bounded and $u_n \rightharpoonup u$. According to Gérard [Gér89, Gér91] there is a subsequence—denoted again by u_n—and a Radon measure μ with values in the positive definite hermitean matrices such that for all $\Psi^0_{\text{comp}}(\Omega, \mathbb{C}^{N\times N})$

$$\langle A(u_n - u), u_n - u\rangle_{L^2} \to \int_{S^*\Omega} a(x,\xi) : \mu(dx\, d\xi).$$

Here a denotes the principal symbol of A. The following Lemma gives the connection to $\text{WF}(u_n)$ in the scalar case.

Lemma 2. *For $N = 1$ there holds*

$$\text{WF}(u_n)|_{S^*\Omega} = \text{supp}\,\mu.$$

Proof. (a) If Au_n is relative compact then

$$\|A(u_n - u)\|^2_{L^2} = \langle A^\dagger A(u_n - u), u_n - u\rangle_{L^2} \to 0,$$

i.e.,

$$\int_{S^*\Omega} |a(x,\xi)|^2 d\mu = 0,$$

which just means that $\text{supp}\,\mu \subset \text{char}\, A$. Thus $\text{supp}\,\mu \subset \text{WF}(u_n)|_{S^*\Omega}$.
(b) Let $(x,\xi) \notin \text{supp}\,\mu$. Choose $a \in S^0$ such that $a(x,\xi) = 1$ but $|a|^2\mu = 0$. Thus $(x,\xi) \notin \text{char}\, A$ and

$$\|A(u_n - u)\|^2_{L^2} = \langle A^\dagger A(u_n - u), u_n - u\rangle_{L^2} \to 0,$$

i.e., Au_n is relatively compact in L^2. Thus, by Lemma 1 $(x,\xi) \notin \text{WF}(u_n)$. □

As a corollary we obtain a simple characterization of the relative compactness of Au_n.

Corollary 1. *Let $A \in \Psi^0$. The sequence Au_n is relative compact iff* $\mathrm{WF}(u_n) \subset \operatorname{char} A$.

Proof. If Au_n is relatively compact we have by (2)

$$\mathrm{WF}(u_n) \subset \mathrm{WF}(Au_n) \cup \operatorname{char} A = \operatorname{char} A.$$

If on the other side $\mathrm{WF}(u_n) \subset \operatorname{char} A$, we have by Lemma 2 for all $\phi \in C_0^\infty(\Omega)$ that

$$|\phi(x)a(x,\xi)|^2\mu = 0,$$

i.e., $\phi A u_n$ converges in L^2. □

3. Polarization Sets

In analogy to the theory of microlocal regularity of vector-valued distributions [Den82], we define for a bounded sequence $u_n \in L^2_{\mathrm{loc}}(\Omega, \mathbb{C}^N)$ the *polarization set*

$$\mathrm{WF}_{\mathrm{pol}}(u_n) = \bigcap_{A \in \Psi^0(\Omega, \mathbb{C}^{1\times N}) : Au_n \text{ rel. compact}} \{(x,\xi;w) \in T^*\Omega \backslash 0 \times \mathbb{C}^N : a(x,\xi).w = 0\}.$$

Lemma 3. *There holds*

$$\mathrm{WF}_{\mathrm{pol}}(u_n) = \bigcap_{A \in \Psi^0(\Omega, \mathbb{C}^{1\times N}) : a.\mu = 0} \{(x,\xi;w) : a(x,\xi).w = 0\},$$

where μ denotes the microlocal defect measure belonging to the sequence u_n.

Proof. For $A \in \Psi^0(\Omega, \mathbb{C}^{1\times N})$ we show that Au_n is relatively compact in $L^2_{\mathrm{loc}}(\Omega)$ iff $a.\mu = 0$ as a vector valued measure. To this end we complete A to a matrix-valued operator

$$A_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \chi(x,D)A \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with only the jth row being non-trivial. The symbol $\chi \in S^0$ is assumed to have compact support in $S^*\Omega$. For all such χ we obtain for Au_n relatively compact by the construction of μ

$$0 = \int_{S^*\Omega} \chi(x,\xi)\, a_j : \mu = \int_{S^*\Omega} \chi(x,\xi)\, (a.\mu)_j.$$

Thus, we conclude $a.\mu = 0$.

On the other hand, if $a.\mu = 0$ we get $a^\dagger a : \mu = 0$, i.e.,

$$\|A(u_n - u)\|^2_{L^2} = \langle A^\dagger A(u_n - u), u_n - u\rangle_{L^2} \to 0,$$

i.e., Au_n is relatively compact. □

In generalization of (2) in §1 we get the following Lemma.

Lemma 4. *For $A \in \Psi^0(\Omega, \mathbb{C}^{M\times N})$ there holds*

$$a(\mathrm{WF}_{\mathrm{pol}}(u_n)) \subset \mathrm{WF}_{\mathrm{pol}}(Au_n),$$

where the principal symbol a operates on the fibre $\mathbb{C}^N$:

$$a(x,\xi;w) = a(x,\xi).w$$

Proof. We proceed in analogy to [Den82, Prop. 2.7, p. 355]. If $(x,\xi;w) \in \mathrm{WF}_{\mathrm{pol}}(u_n)$ we get for all $B \in \Psi^0(\Omega, \mathbb{C}^{1\times M})$ with

$$BAu_n \text{ relatively compact}$$

that $ba.w = 0$, i.e.,

$$(x,\xi; a(x,\xi).w) \in \mathrm{WF}_{\mathrm{pol}}(Au_n).$$

□

4. Compensated Compactness

Using polarization sets the main result of [Gér89] can be rephrased as follows.

Theorem 1. *Let $u_n \rightharpoonup u$ in $L^2_{\mathrm{loc}}(\Omega, \mathbb{C}^N)$ and $Q \in \Psi^0(\Omega, \mathbb{C}^{N\times N})$. If*

$$\langle q(x,\xi).w, w\rangle = 0 \qquad \forall (x,\xi,w) \in \mathrm{WF}_{\mathrm{pol}}(u_n),$$

there holds

$$\langle Qu_n, u_n\rangle \rightharpoonup \langle Qu, u\rangle$$

in the sense of distributions.

This theorem shows that the obstruction to the weak convergence of quadratic functionals is to be found in the microlocal non-compactness of the sequence u_n.

The polarization set can be probed by applying a pseudo-differential operator $P \in \Psi^m(\Omega, \mathbb{C}^{M\times N})$ such that

$$Pu_n \text{ relatively compact in } H^{-m}_{\mathrm{loc}}(\Omega, \mathbb{C}^M).$$

Using the elliptic operator

$$E_{-m} = \begin{pmatrix} (1-\Delta)^{-m/2} & & \\ & \ddots & \\ & & (1-\Delta)^{-m/2} \end{pmatrix} \in \Psi^{-m}(\Omega, \mathbb{C}^M)$$

we obtain that $E_{-m}Pu_n$ is relatively compact in $L^2_{\mathrm{loc}}(\Omega, \mathbb{C}^M)$ and therefore

$$e_{-m}p(\mathrm{WF}_{\mathrm{pol}}(u_n)) \subset T^*\Omega \setminus 0 \times 0,$$

i.e., $(x,\xi;w) \in \mathrm{WF}_{\mathrm{pol}}(u_n)$ implies $(1+|\xi|^2)^{-m/2}p(x,\xi).w = 0$, or equivalently,

$$(x,\xi;w) \in \mathrm{WF}_{\mathrm{pol}}(u_n) \quad \Rightarrow \quad p(x,\xi).w = 0. \tag{3}$$

Summarizing we have proved the following Corollary.

Corollary 2. *Let $u_n \rightharpoonup u$ in $L^2_{\mathrm{loc}}(\Omega, \mathbb{C}^N)$ and $P \in \Psi^m(\Omega, \mathbb{C}^{M\times N})$ such that*

$$Pu_n \text{ relatively compact in } H^{-m}_{\mathrm{loc}}(\Omega, \mathbb{C}^M).$$

If for $Q \in \Psi^0(\Omega, \mathbb{C}^{N\times N})$ the Legendre-Hadamard condition is valid, i.e.,

$$p(x,\xi).w = 0 \quad \Rightarrow \quad \langle q(x,\xi).w, w\rangle = 0,$$

there holds

$$\langle Qu_n, u_n\rangle \rightharpoonup \langle Qu, u\rangle$$

in the sense of distributions.

In the case of real, constant coefficient, first order differential operators

$$Pu = \sum_{j,k} a_{ijk} \frac{\partial u_j}{\partial x_k}, \qquad a_{ijk} \in \mathbb{R},$$

and a real matrix $Q \in \mathbb{R}^{N \times N}$, we recover the results of Tartar's Heriot-Watt lecture notes on compensated compactness [Tar79]. Introducing his Λ-set [Tar79, p. 161],

$$\Lambda = \{w \in \mathbb{R}^N : \exists \xi \neq 0 \text{ such that } \sum_{j,k} a_{ijk} w_j \xi_k = 0\},$$

the implication (3) shows that

$$\mathrm{WF}_{\mathrm{pol}}(u_n) \subset (T^*\Omega \setminus 0) \times (\Lambda + i\Lambda).$$

Therefore, from Theorem 1 the following Corollary easily follows, which is exactly the assertion of [Tar79, Cor. 13, p. 177].

Corollary 3. *Let $u_n \rightharpoonup u$ in $L^2_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and $Q \in \mathbb{R}^{N \times N}$. If*

$$\sum_{j,k} a_{ijk} \frac{\partial u_j}{\partial x_k} \text{ relatively compact in } H^{-1}_{\mathrm{loc}}(\Omega), \; i = 1, \ldots, M,$$

and $\langle Qw, w\rangle = 0$ for all $w \in \Lambda$, there holds

$$\langle Qu_n, u_n\rangle \rightharpoonup \langle Qu, u\rangle$$

in the sense of distributions.

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