Asymptotic independence of the extreme eigenvalues of Gaussian unitary ensemble

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(Received 27 February 2009; accepted 3 December 2009; published online 2 February 2010)

We give a short, operator-theoretic proof of the asymptotic independence (including a first correction term) of the minimal and maximal eigenvalue of the $n\times n$ Gaussian unitary ensemble in the large matrix limit $n\to\infty$. This is done by representing the joint probability distribution of the extreme eigenvalues as the Fredholm determinant of an operator matrix that asymptotically becomes diagonal. As a corollary, we get that the correlation of the extreme eigenvalues asymptotically behaves like $n^{-2/3}/4\sigma^2$, where σ^2 denotes the variance of the Tracy–Widom distribution. While we conjecture that the extreme eigenvalues are asymptotically independent for Wigner random Hermitian matrix ensembles, in general, the actual constant in the asymptotic behavior of the correlation turns out to be specific and can thus be used to distinguish the Gaussian unitary ensemble statistically from certain other Wigner ensembles. © *2010 American Institute of Physics.* [doi:10.1063/1.3290968]

I. INTRODUCTION

We consider the $n\times n$ Gaussian unitary ensemble (GUE) with the joint probability distribution of its (unordered) eigenvalues given by

$$p_n(\lambda_1,\ldots,\lambda_n)=c_n e^{-\lambda_1^2-\cdots-\lambda_n^2}\prod_{i<j}|\lambda_i-\lambda_j|^2$$

and denote the induced minimal and maximal eigenvalue by $\lambda_{\min}^{(n)}$ and $\lambda_{\max}^{(n)}$. Bianchi *et al.* (2008) recently showed the asymptotic independence of the edge-scaled extreme eigenvalues, that is, they proved

$$\mathbb{P}(\tilde{\lambda}_{\min}^{(n)}\le x,\tilde{\lambda}_{\max}^{(n)}\le y)=\mathbb{P}(\tilde{\lambda}_{\min}^{(n)}\le x)\cdot\mathbb{P}(\tilde{\lambda}_{\max}^{(n)}\le y)+o(1)\quad(n\to\infty) \tag{1}$$

with the fluctuations

$$\tilde{\lambda}_{\min}^{(n)}=2^{1/2}n^{1/6}(\lambda_{\min}^{(n)}+\sqrt{2n}),\quad \tilde{\lambda}_{\max}^{(n)}=2^{1/2}n^{1/6}(\lambda_{\max}^{(n)}-\sqrt{2n}). \tag{2}$$

The asymptotic independence can been used (Bianchi *et al.*, 2009) to design, based on the *ratio* of the extreme eigenvalues, a statistical test for the randomness of matrices that does *not* depend on estimating the actual variance of the distribution of the matrix entries (that is, the unknown level of noise in some applications).

In this paper, we shall improve upon these results by showing that the *correlation* of the extreme eigenvalues is a simple, scale-independent device to distinguish the GUE statistically from certain other Wigner random Hermitian matrix ensembles (and not just from nonrandom matrices like the ratio-based test). To this end, we establish a first correction term to the asymptotic independence (1), namely,

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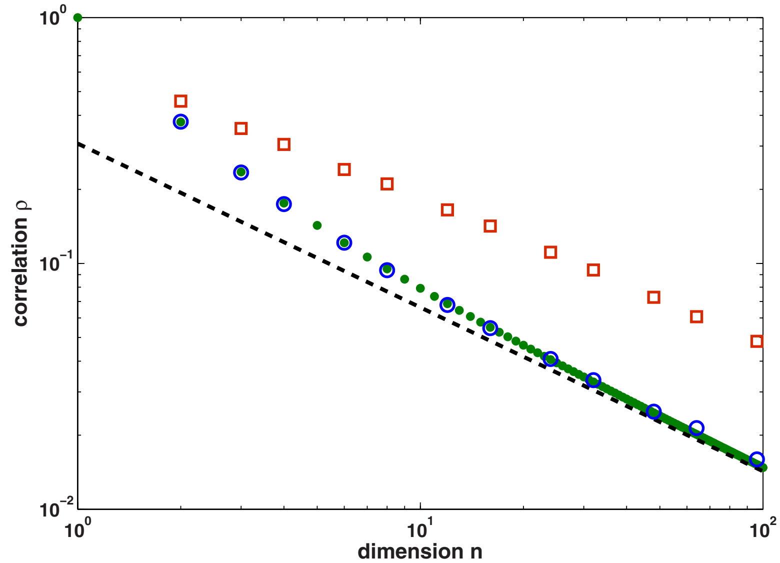

FIG. 1. (Color online) The dots show the values of the correlation ρ of the extreme eigenvalues of the $n\times n$ GUE as obtained from a numerical evaluation of the Fredholm determinant (8) by the method of Bornemann (2009). The dashed line shows the leading order term $n^{-2/3}/4\sigma^2$ of the asymptotic expansion (4). The circles show the sample correlation for 10^6 realizations of $n\times n$ matrices drawn from the GUE. To compare with, the squares show the same for 10^6 realizations of $n\times n$ Hermitian matrices whose algebraic degrees of freedom are uniformly distributed on $[-1,1]$.

$$\mathbb{P}(\tilde{\lambda}_{\min}^{(n)} \le x, \tilde{\lambda}_{\max}^{(n)} \le y) = \mathbb{P}(\tilde{\lambda}_{\min}^{(n)} \le x) \cdot \mathbb{P}(\tilde{\lambda}_{\max}^{(n)} \le y) + \tfrac{1}{4} F_2'(-x) F_2'(y) n^{-2/3} + O(n^{-4/3}) \tag{3}$$

as $n\to\infty$, locally uniform in x and y. Here, F_2 denotes the Tracy–Widom distribution [see (19) below]. In fact, the correction term comes as an additional benefit from a short and conceptually simple new proof of the asymptotic independence that explains it straightforwardly from the asymptotic diagonalization of a certain operator matrix. In contrast, Bianchi *et al.* (2008) based their original proof on quite a detailed and lengthy study of the classical power series of the Fredholm determinants representing the probability distributions in (1).

In Sec. II we discuss the correlation of the extreme eigenvalues. In Sec. III we comment on universality. Finally, in Sec. IV, we prove the expansion (3).

II. THE CORRELATION OF THE EXTREME EIGENVALUES OF GUE

Since both $F_2'(-x)$ and $F_2'(y)$ are probability densities, it follows from (3) that the covariance of the edge-scaled extreme eigenvalues of the GUE satisfies

$$\operatorname{cov}(\tilde{\lambda}_{\min}^{(n)}, \tilde{\lambda}_{\max}^{(n)}) = \tfrac{1}{4} n^{-2/3} + O(n^{-4/3}) \quad (n\to\infty).$$

Therefore, because of scale and shift invariance and by recalling (5) below, we get the correlation of the *unscaled* extreme eigenvalues (or of *any* rescaling thereof) as

$$\rho(\lambda_{\min}^{(n)}, \lambda_{\max}^{(n)}) = \frac{n^{-2/3}}{4\sigma^2} + O(n^{-4/3}) \quad (n\to\infty), \tag{4}$$

where $\sigma^2=0.813\,194\,792\,832\,957\cdots$ is the variance of the Tracy–Widom distribution. Figure 1 visualizes that the leading order term of this expansion is actually quite a precise approximation of the correlation even for rather small dimensions n.

We observe a different asymptotic behavior for random Hermitian matrices whose algebraic degrees of freedom are *uniformly* distributed on $[-1,1]$. Though the data shown in Fig. 1 hint at an asymptotic behavior of the correlation of the form $\rho \simeq cn^{-2/3}$ here too, the constant c is now, quite distinguishably, about three times as large as for the GUE. Therefore, the correlation of the extreme eigenvalues may be used as a simple and effective scale-independent device to distinguish the GUE statistically from certain other Wigner ensembles (as defined in Soshnikov, 1999).

III. UNIVERSALITY

Within the class of Wigner random Hermitian matrix ensembles, there are several limit laws known to hold *universally*. Examples are the universality of the limit eigenvalue density, as given by Wigner's semicircle law, and of the limit distribution of the (properly rescaled) fluctuations of the maximal eigenvalue, as given by the Tracy–Widom distribution (Soshnikov, 1999). It is therefore reasonable to *conjecture* the universality of the asymptotic independence of the extreme eigenvalues. In fact, the sample correlation (squares) shown in Fig. 1 for a concrete non-Gaussian Wigner ensemble strongly points into that direction.

However, since the asymptotic behavior $\rho \simeq cn^{-2/3}$ observed for this example differs in constant c, it appears that the correction term in (3) has to be specific to the GUE. We offer the following explanation for this effect. The Edgeworth expansion (Choup, 2006, 2008) of the largest eigenvalue distribution function of GUE, that is,

$$\mathbb{P}(\tilde{\lambda}_{\max}^{(n)} \le t) = F_2(t) + \gamma(t)n^{-2/3} + O(n^{-1}), \tag{5}$$

where the coefficient $\gamma(t)$ is actually given by an explicit, though quite lengthy expression (Choup, 2008, Theorem 1.3) allows us to infer from (3) a likewise Edgeworth expansion of the joint probability distribution, namely,

$$\begin{aligned}\mathbb{P}(\tilde{\lambda}_{\min}^{(n)} \le x, \tilde{\lambda}_{\max}^{(n)} \le y) &= (1 - F_2(-x)) \cdot F_2(y) \\ &\quad + \big((1 - F_2(-x))\gamma(y) - \gamma(-x)F_2(y) + \tfrac{1}{4}F_2'(-x)F_2'(y)\big)n^{-2/3} + O(n^{-1}).\end{aligned} \tag{6}$$

[Note the considerable amount of cancellation that would have taken place within the order $O(n^{-2/3})$ terms if we had established (3) from those Edgeworth expansions at the first hand.] Though the leading order term $F_2(t)$ of the Edgeworth expansion (5) is known to be universal, the coefficient $\gamma(t)$ of the first correction will, in general, as in the central limit theorem, depend on some higher order moments of the underlying distribution of the matrix entries. Now, since the correction term to the asymptotic independence in (3) is contributing to exactly the same level of approximation in the expansion (6), namely, to the order $O(n^{-2/3})$ term, it will also most likely, in general, depend on the specific probability distribution of the matrix entries.

IV. PROOF OF THE ASYMPTOTIC EXPANSION (3)

A. Step 1: Determinantal representation of the joint probability distribution

Starting point is the well known representation

$$p_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{n!}\det(K_n(\lambda_i, \lambda_j))_{i,j=1}^n$$

of the joint eigenvalue distribution in terms of the finite rank kernel

$$K_n(\xi, \eta) = \sum_{k=0}^{n-1} \phi_k(\xi)\phi_k(\eta) = \frac{1}{2}\frac{\phi_n(\xi)\phi_n'(\eta) - \phi_n'(\xi)\phi_n(\eta)}{\xi - \eta} - \frac{1}{2}\phi_n(\xi)\phi_n(\eta) \tag{7}$$

(the second equality follows from the Christoffel–Darboux formula) that is built from the $L^2(\mathbb{R})$-orthonormal system of the Hermite functions

$$\phi_m(t) = \frac{e^{-t^2/2}H_m(t)}{\pi^{1/4}\sqrt{m!}2^{m/2}}.$$

From this representation, we get the determinantal formulas (Deift, 1999, Sec. 5.4)

$$\mathbb{P}(X \le \lambda_{\min}^{(n)}) = \det(I - K_n\restriction_{L^2(-\infty,X)}), \quad \mathbb{P}(\lambda_{\max}^{(n)} \le Y) = \det(I - K_n\restriction_{L^2(Y,\infty)}),$$

$$\mathbb{P}(X \le \lambda_{\min}^{(n)}, \lambda_{\max}^{(n)} \le Y) = \det(I - K_n\restriction_{L^2((-\infty,X)\cup(Y,\infty))})$$

with the natural constraint $X < Y$ (otherwise the last probability would be zero). [Given a kernel $K(\xi,\eta)$, we denote by $K\restriction_{\mathcal{H}}:\mathcal{H}\to\mathcal{H}$ the induced integral operator on a Hilbert space $\mathcal{H}$ of functions.]

B. Step 2: Representation in terms of operator-matrix determinants

While Bianchi *et al.* (2008) discussed exactly these determinants using Fredholm's power series, we refer to the fact that, for $X < Y$,

$$L^2((-\infty,X) \cup (Y,\infty)) = L^2(-\infty,X) \oplus L^2(Y,\infty)$$

by means of the isometric isomorphism

$$u \in L^2((-\infty,X) \cup (Y,\infty)) \mapsto (u|_{(-\infty,X)}, u|_{(Y,\infty)}) \in L^2(-\infty,X) \oplus L^2(Y,\infty).$$

This implies (Gohberg *et al.*, 2000, Theorem VI.6.1) the equivalent representation

$$\mathbb{P}(X \le \lambda_{\min}^{(n)}, \lambda_{\max}^{(n)} \le Y) = \det\left(I - \begin{pmatrix} K_n & K_n \\ K_n & K_n \end{pmatrix}\restriction_{L^2(-\infty,X)\oplus L^2(Y,\infty)}\right) \tag{8}$$

in terms of an operator matrix, as can be understood from the following argument (Bornemann, 2008, Sec. 8.1): given a function $u \in L^2((-\infty,X)\cup(Y,\infty))$, the operational expression $v = K_n\restriction_{L^2((-\infty,X)\cup(Y,\infty))}u$ is just the short-hand notation for

$$\begin{aligned} v(\xi) &= \int_{(-\infty,X)\cup(Y,\infty)} K_n(\xi,\eta)u(\eta)\,\mathrm{d}\eta \\ &= \int_{-\infty}^{X} K_n(\xi,\eta)u(\eta)\,\mathrm{d}\eta + \int_{Y}^{\infty} K_n(\xi,\eta)u(\eta)\,\mathrm{d}\eta \quad (\xi \in (-\infty,X) \cup (Y,\infty)), \end{aligned}$$

which, by writing $u_1 = u|_{(-\infty,X)}$, $u_2 = u|_{(Y,\infty)}$ and likewise for v, can be rewritten as

$$v_1(\xi) = \int_{-\infty}^{X} K_n(\xi,\eta)u_1(\eta)\,\mathrm{d}\eta + \int_{Y}^{\infty} K_n(\xi,\eta)u_2(\eta)\,\mathrm{d}\eta \quad (\xi \in (-\infty,X)),$$

$$v_2(\xi) = \int_{-\infty}^{X} K_n(\xi,\eta)u_1(\eta)\,\mathrm{d}\eta + \int_{Y}^{\infty} K_n(\xi,\eta)u_2(\eta)\,\mathrm{d}\eta \quad (\xi \in (Y,\infty)).$$

However, the last two equations are just the long-hand notation for the operator-matrix expression

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} K_n & K_n \\ K_n & K_n \end{pmatrix}\restriction_{L^2(-\infty,X)\oplus L^2(Y,\infty)}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

C. Step 3: Edge scaling

Now, the edge scaling (2) amounts for the transformation

$$X = -\sqrt{2n} - 2^{-1/2}n^{-1/6}x, \quad Y = \sqrt{2n} + 2^{-1/2}n^{-1/6}y$$

of the variables X and Y; a substitution into the corresponding integral expression transforms the kernel entries of (8) into

$$K_{11}^{(n)}(\xi,\eta) = 2^{-1/2}n^{-1/6}K_n(-\sqrt{2n} - 2^{-1/2}n^{-1/6}\xi, -\sqrt{2n} - 2^{-1/2}n^{-1/6}\eta),$$

$$K_{12}^{(n)}(\xi,\eta) = 2^{-1/2}n^{-1/6}K_n(-\sqrt{2n} - 2^{-1/2}n^{-1/6}\xi, \sqrt{2n} + 2^{-1/2}n^{-1/6}\eta),$$

$$K_{21}^{(n)}(\xi,\eta) = 2^{-1/2}n^{-1/6}K_n(\sqrt{2n} + 2^{-1/2}n^{-1/6}\xi, -\sqrt{2n} - 2^{-1/2}n^{-1/6}\eta),$$

$$K_{22}^{(n)}(\xi,\eta) = 2^{-1/2}n^{-1/6}K_n(\sqrt{2n} + 2^{-1/2}n^{-1/6}\xi, \sqrt{2n} + 2^{-1/2}n^{-1/6}\eta).$$

For instance, the upper left entry of the operator matrix in (8), which acts from $L^2(-\infty,X)$ into itself, is defined by the integral

$$v(\hat{\xi}) = \int_{-\infty}^{X} K_n(\hat{\xi},\hat{\eta})u(\hat{\eta})\,\mathrm{d}\hat{\eta} \quad (\hat{\xi} \in (-\infty,X))$$

that transforms by means of the substitutions $\hat{u}(\eta)=u(\hat{\eta})$, $\hat{v}(\xi)=v(\hat{\xi})$, and

$$\hat{\xi} = -\sqrt{2n} - 2^{-1/2}n^{-1/6}\xi, \quad \hat{\eta} = -\sqrt{2n} - 2^{-1/2}n^{-1/6}\eta,$$

into the integral

$$\hat{v}(\xi) = \int_{x}^{\infty} K_{11}^{(n)}(\xi,\eta)\hat{u}(\eta)\,\mathrm{d}\eta \quad (\xi \in (x,\infty))$$

with kernel $K_{11}^{(n)}$, inducing an integral operator that acts from $L^2(x,\infty)$ into itself. The other three entries of the operator matrix are dealt with similarly.

From (8), we thus obtain (note that $X<Y$ eventually for n large if x and y stay bounded) the representation

$$\mathbb{P}(-\tilde{\lambda}_{\min}^{(n)} \le x) = \det(I - K_{11}^{(n)}\restriction_{L^2(x,\infty)}),$$

$$\mathbb{P}(\tilde{\lambda}_{\max}^{(n)} \le y) = \det(I - K_{22}^{(n)}\restriction_{L^2(y,\infty)}),$$

$$\mathbb{P}(-\tilde{\lambda}_{\min}^{(n)} \le x, \tilde{\lambda}_{\max}^{(n)} \le y) = \det\left(I - \begin{pmatrix} K_{11}^{(n)} & K_{12}^{(n)} \\ K_{21}^{(n)} & K_{22}^{(n)} \end{pmatrix}\restriction_{L^2(x,\infty)\oplus L^2(y,\infty)}\right).$$

The rest of our proof gets clearer if we fix the underlying Hilbert space independently of x and y. To this end, we introduce the orthonormal projection $P_t: L^2(\mathbb{R}) \to L^2(t,\infty)$, which is simply given by the multiplication operator with the characteristic function χ_t of (t,∞). Extending an integral operator $T: L^2(x,\infty) \to L^2(y,\infty)$ by zero into the orthogonal complements of its domain and range, that is, looking at the operator

$$P_y T P_x : L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

finally yields the following determinantal representation that is particularly well suited for an asymptotic treatment:

$$\mathbb{P}(-\tilde{\lambda}_{\min}^{(n)} \le x) = \det(I - P_x K_{11}^{(n)} P_x), \quad \mathbb{P}(\tilde{\lambda}_{\max}^{(n)} \le y) = \det(I - P_y K_{22}^{(n)} P_y),$$

$$\mathbb{P}(-\tilde{\lambda}_{\min}^{(n)} \le x, \tilde{\lambda}_{\max}^{(n)} \le y) = \det\left(I - \begin{pmatrix} P_x K_{11}^{(n)} P_x & P_x K_{12}^{(n)} P_y \\ P_y K_{21}^{(n)} P_x & P_y K_{22}^{(n)} P_y \end{pmatrix}\right). \tag{9}$$

Here, the operator matrix acts on the space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

D. Step 4: The leading order term of the asymptotic expansion

If we plug the Plancherel–Rotach expansion (Szegő, 1975, Theorem 8.22.9) of the Hermite functions, that is, the locally uniform expansion

$$\phi_n(\sqrt{2n}+2^{-1/2}n^{-1/6}t) = 2^{1/4}n^{-1/12}\Big(\mathrm{Ai}(t) - \tfrac{1}{2}\mathrm{Ai}'(t)n^{-1/3} + O(n^{-2/3})\Big),$$

into the rightmost expression defining the kernel K_n in (7), we obtain the locally uniform asymptotic expansion

$$K_{11}^{(n)}(\xi,\eta) = K_{22}^{(n)}(\xi,\eta) = K(\xi,\eta) + O(n^{-2/3}) \quad (n\to\infty) \tag{10}$$

with the Airy kernel

$$K(\xi,\eta) = \frac{\mathrm{Ai}(\xi)\mathrm{Ai}'(\eta) - \mathrm{Ai}'(\xi)\mathrm{Ai}(\eta)}{\xi-\eta}.$$

Furthermore, this expansion of the kernels implies (Choup, 2006, Theorem 1.2) the asymptotic expansion

$$P_tK_{11}^{(n)}P_t = P_tK_{22}^{(n)}P_t = P_tKP_t + O(n^{-2/3}) \quad (n\to\infty) \tag{11}$$

of the induced trace class operators [that is, the error $O(n^{-2/3})$ is also valid in trace norm]. Completely analogously, by using the Plancherel–Rotach expansion once more and recalling the symmetry $\phi_n(-t)=(-1)^n\phi_n(t)$, we obtain the locally uniform expansion

$$K_{12}^{(n)}(\xi,\eta) = K_{21}^{(n)}(\xi,\eta) = \tfrac{1}{2}(-1)^{n-1}\mathrm{Ai}(\xi)\mathrm{Ai}(\eta)n^{-1/3} + O(n^{-1}) \quad (n\to\infty), \tag{12}$$

which, by the same arguments as Choup's, implies the validity of the asymptotic expansion

$$P_tK_{12}^{(n)}P_s = P_tK_{21}^{(n)}P_s = \tfrac{1}{2}(-1)^{n-1}(\chi_t\mathrm{Ai}\otimes\mathrm{Ai}\chi_s)n^{-1/3} + O(n^{-1}) \quad (n\to\infty) \tag{13}$$

of the induced trace class operators. This shows that the off-diagonal operators in (9) have trace norm $O(n^{-1/3})$. [Basically, (11) is derived from (10) in two steps (Choup, 2006, pp. 7–8): first, he observed that the terms of the asymptotic expansion of the kernel $K_{11}(\xi,\eta)$ are finite linear combinations of products of polynomials in ξ and η with one of the following terms: $\mathrm{Ai}(\xi)\mathrm{Ai}(\eta)$, $\mathrm{Ai}'(\xi)\mathrm{Ai}(\eta)$, $\mathrm{Ai}(\xi)\mathrm{Ai}'(\eta)$, $\mathrm{Ai}'(\xi)\mathrm{Ai}'(\eta)$, or $K(\xi,\eta)$. Second, because the Hermite and Airy functions (and their derivatives) decay superexponentially at $+\infty$, all the terms of the asymptotic expansions are trace class operators on $L^2(t,\infty)$. As can be read off from (12), the first step of the argument can easily be extended to $K_{12}(\xi,\eta)$; then, the second step applies *verbatim* to finally prove (13).] Thus, by the local Lipschitz continuity of the determinant with respect to the trace norm (Simon, 2005, Theorem 3.4) we get

$$\begin{aligned}\mathbb{P}(-\tilde{\lambda}_{\min}^{(n)} \le x, \tilde{\lambda}_{\max}^{(n)} \le y) &= \det\left(I - \begin{pmatrix} P_xK_{11}^{(n)}P_x & 0 \\ 0 & P_yK_{22}^{(n)}P_y\end{pmatrix}\right) + O(n^{-1/3}) \\ &= \det(I - P_xK_{11}^{(n)}P_x)\cdot\det(I-P_yK_{22}^{(n)}P_y) + O(n^{-1/3}) \\ &= \mathbb{P}(-\tilde{\lambda}_{\min}^{(n)} \le x)\cdot\mathbb{P}(\tilde{\lambda}_{\max}^{(n)} \le y) + O(n^{-1/3}),\end{aligned}$$

where we have used the multiplication rule of the determinant for *diagonal* operator matrices. This proves the asymptotic independence result (1) of Bianchi *et al.* (2008); here with an additional estimate of the order of approximation, however.

E. Step 5: The first correction term of the asymptotic expansion

We start with the operator-matrix factorization

$$I - \begin{pmatrix} P_x K_{11}^{(n)} P_x & P_x K_{12}^{(n)} P_y \\ P_y K_{21}^{(n)} P_x & P_y K_{22}^{(n)} P_y \end{pmatrix} = \left(I - \begin{pmatrix} P_x K_{11}^{(n)} P_x & 0 \\ 0 & P_y K_{22}^{(n)} P_y \end{pmatrix} \right)$$
$$\cdot \left(I - \begin{pmatrix} 0 & (I - P_x K_{11}^{(n)} P_x)^{-1} P_x K_{12}^{(n)} P_y \\ (I - P_y K_{22}^{(n)} P_y)^{-1} P_y K_{21}^{(n)} P_x & 0 \end{pmatrix} \right),$$

which exists since the operators $I - P_x K_{11}^{(n)} P_x$ and $I - P_y K_{22}^{(n)} P_y$ are invertible having determinants that, by the first two equations of (9), represent *nonvanishing* probabilities. If we plug this factorization into the third equation of (9), we obtain

$$\mathbb{P}(-\tilde{\lambda}_{\min}^{(n)} \le x, \tilde{\lambda}_{\max}^{(n)} \le y) = \det\left(I - \begin{pmatrix} P_x K_{11}^{(n)} P_x & 0 \\ 0 & P_y K_{22}^{(n)} P_y \end{pmatrix} \right)$$
$$\cdot \det\left(I - \begin{pmatrix} 0 & (I - P_x K_{11}^{(n)} P_x)^{-1} P_x K_{12}^{(n)} P_y \\ (I - P_y K_{22}^{(n)} P_y)^{-1} P_y K_{21}^{(n)} P_x & 0 \end{pmatrix} \right). \tag{14}$$

As above, the first determinantal term of this product evaluates to

$$\det(I - P_x K_{11}^{(n)} P_x) \cdot \det(I - P_y K_{22}^{(n)} P_y) = \mathbb{P}(-\tilde{\lambda}_{\min}^{(n)} \le x) \cdot \mathbb{P}(\tilde{\lambda}_{\max}^{(n)} \le y). \tag{15}$$

By (11) and (13), the second determinantal term of (14) can be written briefly as

$$\det\left(I - \frac{(-1)^{n-1} n^{-1/3}}{2} \begin{pmatrix} 0 & T_{xy} + O(n^{-2/3}) \\ T_{yx} + O(n^{-2/3}) & 0 \end{pmatrix} \right) \tag{16}$$

with the rank-one operator

$$T_{ts} = (I - P_t K P_t)^{-1} \chi_t \ \mathrm{Ai} \otimes \mathrm{Ai} \ \ \chi_s.$$

For trace class operators A and B, the Schur decomposition [Gohberg *et al.*, 2000, (I.3.8)]

$$\begin{pmatrix} I & A \\ B & I \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I - AB & 0 \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}$$

implies the determinantal formula

$$\det\left(I + \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \right) = \det(I - AB)$$

the determinant (16) evaluates and expands to

$$\det\left(I - \tfrac{1}{4} n^{-2/3} (T_{xy} + O(n^{-2/3}))(T_{yx} + O(n^{-2/3})) \right) = \det\left(I - \tfrac{1}{4} n^{-2/3} T_{xy} T_{yx} \right) + O(n^{-4/3}).$$

This last determinant can actually be evaluated *exactly*. Indeed, by using the fact that $(f \otimes g)(v \otimes w) = \langle g, v \rangle f \otimes w$, and thus

$$\det(I - (f \otimes g)(v \otimes w)) = \det(I - \langle g, v \rangle f \otimes w) = 1 - \langle g, v \rangle \langle f, w \rangle,$$

we obtain

$$\det\left(I - \tfrac{1}{4} n^{-2/3} T_{xy} T_{yx} \right) = 1 - \tfrac{1}{4} u(x) u(y) n^{-2/3}$$

with the function

$$u(t) = \langle (I - P_t K P_t)^{-1} \chi_t \ \mathrm{Ai}, \mathrm{Ai} \ \ \chi_t \rangle. \tag{17}$$

To summarize, our result so far is

$$\mathbb{P}(-\tilde{\lambda}_{\min}^{(n)} \le x, \tilde{\lambda}_{\max}^{(n)} \le y) = \mathbb{P}(-\tilde{\lambda}_{\min}^{(n)} \le x)\mathbb{P}(\tilde{\lambda}_{\max}^{(n)} \le y) \cdot \left(1 - \tfrac{1}{4}u(x)u(y)n^{-2/3} + O(n^{-4/3})\right).$$

The product of the probabilities with the term $u(x)u(y)$ can further be simplified by expanding it, written as the determinantal expression in (15), through

$$\det(I - P_t K_{11}^{(n)} P_t) = \det(I - P_t K_{22}^{(n)} P_t) = \det(I - P_t K P_t) + O(n^{-2/3}). \tag{18}$$

Now, by introducing the Tracy–Widom distribution

$$F_2(t) = \det(I - P_t K P_t), \tag{19}$$

and by recalling that its logarithmic derivative is actually the function $u(t)$ defined in (17), that is, by recalling the formula $u(t)=F_2'(t)/F_2(t)$, we finally get

$$\mathbb{P}(-\tilde{\lambda}_{\min}^{(n)} \le x, \tilde{\lambda}_{\max}^{(n)} \le y) = \mathbb{P}(-\tilde{\lambda}_{\min}^{(n)} \le x) \cdot \mathbb{P}(\tilde{\lambda}_{\max}^{(n)} \le y) - \tfrac{1}{4}F_2'(x)F_2'(y)n^{-2/3} + O(n^{-4/3}),$$

which is easily seen to be equivalent to the asserted expansion (3). [The above mentioned formula for $u(t)$ was implicitly obtained in the course of Tracy and Widom's (1994) original derivation of their famous Painlevé II representation of F_2; it straightforwardly follows from Eqs. (1.1), (2.4), and (2.21) of their paper. The formula is stated explicitly in Widom (2004, p. 1132), however.]

1. Remark

Numerical experiments using the methods of Bornemann (2008) show that the error term of this expansion is indeed not better than of the order of $O(n^{-4/3})$.

ACKNOWLEDGMENTS

The author thanks Herbert Spohn for helpful comments on the first draft and for bringing up the issue of universality.

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