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REGULAR ARTICLE

Finite-element discretization of static Hamilton-Jacobi equations based on a local variational principle

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Abstract We propose a linear finite-element discretization of Dirichlet problems for static Hamilton–Jacobi equations on unstructured triangulations. The discretization is based on simplified localized Dirichlet problems that are solved by a local variational principle. It generalizes several approaches known in the literature and allows for a simple and transparent convergence theory. In this paper the resulting system of nonlinear equations is solved by an *adaptive* Gauss–Seidel iteration that is easily implemented and quite effective as a couple of numerical experiments show.

Keywords Hamilton–Jacobi equation · Linear finite elements · Local variational principle · Viscosity solutions · Compatibility condition · Hopf–Lax formula · Eikonal equation · Adaptive Gauss–Seidel iteration

1 Introduction

With the advent (Osher and Sethian 1988) and success of level set methods and its many applications (Osher and Fedkiw 2003; Sethian 1999) to areas ranging from computational physics to computer vision there has been considerable interest in numerical methods for solving Hamilton–Jacobi equations, dynamic and static. For problems with complex geometries, or for problems on manifolds, there is a demand for methods that work on unstructured meshes such as triangulations.

Three main directions of constructing discretizations on unstructured meshes can be found in the literature. First, there are methods that lift ideas of finite-difference upwinding and Godunov schemes from hyperbolic conservation laws to Hamilton–Jacobi equations (whose solutions are, at least in 1D, *integrals* of solutions to conservation laws), see, e.g., Barth and Sethian (1998). Second, there are finite-element methods that are based on a weak formulation of the semi-linear second order equation obtained by adding a small amount of factual viscosity, see, e.g., Li et al. (2003). And third, there are methods that utilize the connection of Hamilton–Jacobi equations (via Bellman's principle) to optimal control problems, see, e.g., Sethian and Vladimirsky (2003).

In this paper we propose a discretization that bears similarities with the last approach. We implicitly construct a linear finite-element solution by requiring that it solves *locally* a *simplified* equation with (local) boundary conditions given by the finite-element function itself. The simplified local equation is then solved by a *local variational principle*, the *Hopf–Lax formula*.

This simple discretization is interesting in various respects. First, we will show that it generalizes quite a few approaches known in the literature. Second, it allows for an extremely simple, self-contained convergence theory. In fact, the only results of the general theory that we rely on are a uniqueness theorem for viscosity solutions and the theorem of Arzelà–Ascoli. Existence of viscosity

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Dedicated to Peter Deuflhard on the occasion of his 60th birthday.

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solutions will be shown *in passing* by the convergence of the finite-element discretization.

By construction the discretization inherits structural properties of the viscosity solution of the Hamilton–Jacobi equation such as a comparison principle. For each property we will carefully trace the specific assumptions on the Hamiltonian and the boundary data that are needed for proofs in the continuous and the discrete case. In particular, it is known (Bardi and Capuzzo -Dolcetta 1997; Lions 1982) that the existence of a viscosity solution of the Dirichlet problem necessitates a *compatibility condition* on the boundary data, which is basically a restrictive Lipschitz bound. However, this necessary condition gets barely any mention in the literature on numerical methods, even in the formulation of convergence results, e.g., Sethian and Vladimirsky (2003, Theorem 7.7).

Moreover, we propose in this paper a likewise simple iterative method for solving the resulting nonlinear system of equations, namely the *adaptive Gauss–Seidel iteration*, which is easily implemented and, at least experimentally, quite effective.

The paper is organized as follows. In Sect. 2 we recall the concept, existence, and uniqueness of viscosity solutions of Dirichlet problems for certain Hamilton–Jacobi equations. In Sect. 3 we introduce the support function of the zero-level set of the Hamiltonian which plays a major role in the definition of the discretization. In Sect. 4 we define the linear finite-element solution based on a local variational principle. The existence, uniqueness, and uniform Lipschitz continuity of the finite-element solutions are subject of Sect. 5. A suitable concept of *consistency* is introduced in Sect. 6 and the convergence of the discrete solutions is proved. In Sect. 7 we apply the finite-element discretization to a class of generalized eikonal equations in 2D and obtain, by a simple geometric argument, a closed formula for the local discrete equation. In Sect. 8 we discuss the adaptive Gauss–Seidel iteration that we propose for an easily implemented and quite effective solution of the nonlinear system of equations. Finally, in Sect. 9 we study two numerical experiments and compare the proposed method to the ordered upwind method (OUM) recently published by Sethian and Vladimirsky (2003).

2 Existence and uniqueness of viscosity solutions

In this section we shortly review the existence and uniqueness theory for the Dirichlet problem of a Hamilton–Jacobi equation,

$$H(x, Du(x)) = 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = g, \tag{1}$$

where throughout the paper we will assume that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. For convex Hamiltonians H a sufficiently general set of assumptions is (see Lions 1982, Sect. 5.2):

(H1) (Continuity) $H \in C(\overline{\Omega} \times \mathbb{R}^d)$.
(H2) (Convexity) $p \mapsto H(x,p)$ is convex for all $x \in \overline{\Omega}$.
(H3) (Coercivity) $H(x,p) \to \infty$ as $\|p\| \to \infty$, uniformly in $x \in \overline{\Omega}$. Equivalently, by assumptions (H1) and (H2), there are positive constants α, β with

$$H(x,p) \geqslant \alpha\|p\| - \beta, \quad x \in \overline{\Omega}, \, p \in \mathbb{R}^d.$$

(H4) (Compatibility of the Hamiltonian) $H(x, 0) \leqslant 0$ for all $x \in \overline{\Omega}$.

Existence of a solution to (1) requires a further condition on the boundary data:

(H5) (Compatibility of Dirichlet data) $g(x) - g(y) \leqslant \delta(x, y)$ for all $x, y \in \partial\Omega$.

Here, δ denotes the *optical distance* defined, under the assumptions (H1)–(H4), by

$$\delta(x,y) = \inf\left\{ \int_0^1 \rho\big(\xi(t), -\xi'(t)\big)\,\mathrm{d}t \;:\; \xi \in C^{0,1}([0,1], \overline{\Omega}), \; \xi(0) = x, \; \xi(1) = y \right\}$$

$$\text{where} \quad \rho(x,q) = \max_{H(x,p)=0} \langle p, q\rangle\,. \tag{2}$$

In fact, δ qualifies as a distance by the fairly obvious properties

$$\delta(x,x) = 0, \qquad 0 \leqslant \delta(x,z) \leqslant \delta(x,y) + \delta(y,z), \quad x,y,z \in \overline{\Omega}. \tag{3}$$

If H is symmetric with respect to p, that is, $H(x,p) = H(x,-p)$, then δ defines a pseudometric on $\overline{\Omega}$.

Let us recall the concept of viscosity solutions (Crandall et al. 1984) for the first order equation

$$H(x, Du(x)) = 0, \quad x \in \Omega. \tag{4}$$

A function $u \in C^{0,1}(\overline{\Omega})$ is a *viscosity subsolution (supersolution)* of (4) if all $v \in C_0^\infty(\Omega)$ with $u - v$ attaining a local maximum (minimum) at some $x_0 \in \Omega$ yield

$$H(x_0, Dv(x_0)) \leqslant 0 \quad (\geqslant 0).$$

Now, a *viscosity solution* is simultaneously a viscosity sub- and supersolution. Note that by Rademacher's theorem on the differentiability of Lipschitz continuous functions a viscosity solution satisfies (4) pointwise almost everywhere.

Theorem 1 *(*Lions 1982, Theorem 5.3*) Assume (H1)–(H4). The Dirichlet problem (1) has a viscosity solution u if and only if the boundary condition satisfies the compatibility condition (H5). A specific viscosity solution is then given by the Hopf–Lax formula*

$$u(x) = \inf_{y\in\partial\Omega} \Big(g(y) + \delta(x,y)\Big). \tag{5}$$

While this theorem will only serve as a motivation for the finite-element discretization in Sect. 5, we will obtain the existence of viscosity solutions under somewhat more restrictive assumptions as a spin-off of the convergence result, Theorem 11.

Uniqueness requires a compatibility condition on the Hamiltonian that is slightly stronger than (H4):

$(H4')$ $H(x,0) < 0$ for all $x \in \Omega$.

In fact, uniqueness of the viscosity solution given by (5) is then a simple corollary of the following *comparison principle*.

Theorem 2 *(*Ishii 1987*) Assume (H1)–(H3) and (H4′). Let u, v be viscosity sub- and supersolutions of (4), respectively. If $u \leqslant v$ on $\partial\Omega$ then $u \leqslant v$ on $\overline{\Omega}$.*

3 The support function of the zero-level set

In the definition (2) of the optical distance δ the *support function* (see Rockafellar 1970, p. 28)

$$\rho(x,q) = \max_{H(x,p)=0} \langle p,q\rangle = \sup_{H(x,p)\leqslant 0} \langle p,q\rangle, \quad x\in\overline{\Omega},\ q\in\mathbb{R}^d, \tag{6}$$

of the zero-level set of H made an appearance. It is a well-defined real-valued function, since by (H3) the zero-level set is compact and by (H4) non-empty. The second equality in (6) follows from the convexity (H2).

Since the discretization that we propose in the next section will be based on this support function ρ we collect its most important properties.

Lemma 3 *Assume (H1)–(H4). Then $\rho : \overline{\Omega}\times\mathbb{R}^d \to \mathbb{R}$ is upper semicontinuous in the first argument, positively homogeneous convex in the second, that is,*

$$\rho(x,q_1+q_2) \leqslant \rho(x,q_1)+\rho(x,q_2), \quad \rho(x,tq)=t\rho(x,q), \quad t\geqslant 0,$$

for $x \in \overline{\Omega}$ and $q, q_1, q_2 \in \mathbb{R}^d$. Let $\rho^* = \beta/\alpha$ with α, β from (H4). Then

$$0 \leqslant \rho(x,q) \leqslant \rho^* \|q\|, \quad x\in\overline{\Omega},\ q\in\mathbb{R}^d.$$

Assume additionally (H4′). Then $\rho|_{\Omega\times\mathbb{R}^d}$ is continuous and

$$\rho(x,q) > 0, \quad x\in\Omega,\ 0\neq q\in\mathbb{R}^d.$$

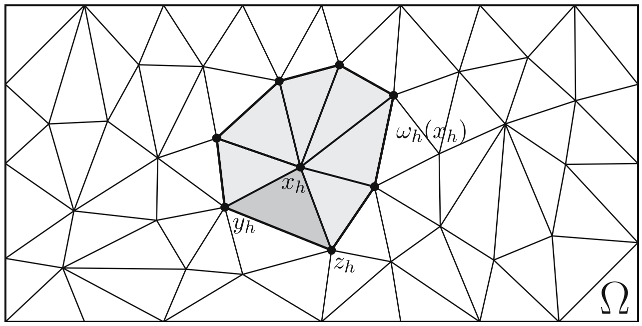

Fig. 1 The neighborhood $\omega_h(x_h)$ of $x_h \in \Omega_h$, that is, the collection of all simplices (*like the one shaded in dark*) that have x_h as a vertex

Proof Being defined as the pointwise supremum of linear functions the function $q \mapsto \rho(x,q)$ is a convex, positively homogeneous and, by (H4), nonnegative function for fixed $x \in \overline{\Omega}$. Assumption (H3) yields for $H(x,p) \leqslant 0$ the bound $\|p\| \leqslant \beta/\alpha = \rho^*$, which readily implies the upper bound on ρ.

To prove the upper semicontinuity let $x_n \to x_0 \in \overline{\Omega}$ and $q \in \mathbb{R}^d$. We extract a subsequence $x_{n'}$ such that

$$\rho(x_{n'},q) = \langle p_{n'},q\rangle \to \limsup_{n\to\infty} \rho(x_n,q),$$

where $p_{n'}$ is a maximizing argument with $H(x_{n'},p_{n'}) = 0$. Because of the bound $\|p_{n'}\| \leqslant \rho^*$ we can assume without loss of generality that $p_{n'} \to p_0$. We obtain $H(x_0,p_0) = 0$ and therefore

$$\limsup_{n\to\infty} \rho(x_n,q) = \langle p_0,q\rangle \leqslant \rho(x_0,q).$$

From now on, we assume (H4′). Let $x \in \Omega$. Since $H(x,0) < 0$ there is a $\delta > 0$ such that $H(x,p) \leqslant 0$ for $\|p\| \leqslant \delta$. Thus, for $q \neq 0$

$$\rho(x,q) \geqslant \max_{\|p\|\leqslant\delta} \langle p,q\rangle = \delta\,\|q\| > 0.$$

Finally, to prove the lower semicontinuity let $x_n \to x_0 \in \Omega$ and $q \in \mathbb{R}^d$. There is a maximizing $p_0 \in \mathbb{R}^d$ with $\rho(x_0,q) = \langle p_0,q\rangle$ and $H(x_0,p_0) = 0$. We extract a subsequence such that $\rho(x_{n'},q) \to \liminf_{n\to\infty} \rho(x_n,q)$ and, below, construct a sequence $p_{n'} \to p_0$ with $H(x_{n'},p_{n'}) \leqslant 0$. With it in hand we conclude

$$\liminf_{n\to\infty} \rho(x_n,q) = \lim_{n'\to\infty} \rho(x_{n'},q) \geqslant \lim_{n'\to\infty} \langle p_{n'},q\rangle = \langle p_0,q\rangle = \rho(x_0,q).$$

There is no loss of generality in assuming that either always $H(x_{n'},p_0) \leqslant 0$ or always $H(x_{n'},p_0) > 0$. In the first case we simply take $p_{n'} = p_0$. In the second case, since $H(x_{n'},0) < 0$, there is a $\lambda_{n'} \in (0,1)$ with $H(x_{n'},\lambda_{n'}p_0) = 0$ and we put $p_{n'} = \lambda_{n'}p_0$. We can assume that $\lambda_{n'} \to \lambda_0 \in [0,1]$. Taking limits in

$$0 = H(x_{n'},\lambda_{n'}p_0) \leqslant (1-\lambda_{n'})H(x_{n'},0) + \lambda_{n'}H(x_{n'},p_0)$$

yields $0 \leqslant (1-\lambda_0)H(x_0,0)$ which, by (H4′), implies $\lambda_0 = 1$ and $p_{n'} \to p_0$. □

Note that if (H4′) holds and H is symmetric with respect to p, then $q \mapsto \rho(x,q)$ defines a norm on $\mathbb{R}^d$ for all $x \in \Omega$.

Lemma 4 *Assume (H1)–(H4) and that the segment joining the points $y, z \in \overline{\Omega}$ belongs to $\overline{\Omega}$. Let $\rho_* \geqslant 0$ be a constant such that $\rho(x,q) \geqslant \rho_* \|q\|$, $x \in \Omega$ and $q \in \mathbb{R}^d$. Then, with the constant ρ^* defined in Lemma 3,*

$$\rho_* \|y-z\| \leqslant \delta(y,z) \leqslant \rho^* \|y-z\|.$$

If $H(x,p)$ does not depend on x, then $\rho(x,q) = \rho(q)$ does not depend on x either and

$$\delta(y,z) = \rho(y-z).$$

Proof The optical distance $\delta(y,z)$ is bounded by the expressions

$$\rho_0 \cdot \inf \left\{ \int_0^1 \|\xi'(t)\| \, \mathrm{d}t : \quad \xi \in C^{0,1}([0,1], \overline{\Omega}) \text{ such that } \xi(0) = y,\ \xi(1) = z \right\},$$

with $\rho_0 = \rho_*$ for the lower bound and $\rho_0 = \rho^*$ for the upper bound. The infimum is nothing but the minimal length of a path joining the point y and z *within* $\overline{\Omega}$. By the assumption on y and z this minimum is realized by the segment joining them.

If H, and hence ρ, does not depend on $x \in \overline{\Omega}$, we obtain by Jensen's inequality for $\xi \in C^{0,1}([0,1], \overline{\Omega})$ with $\xi(0) = y$ and $\xi(1) = z$ that

$$\int_0^1 \rho(-\xi'(t)) \, \mathrm{d}t \geqslant \rho\left(- \int_0^1 \xi'(t) \, \mathrm{d}t \right) = \rho(y-z).$$

The lower bound is attained for the segment joining y and z yielding the assertion $\delta(y,z) = \rho(y-z)$ (see also Lions 1982, Remark 5.7). □

Example 1 An important class[1] of Hamiltonians satisfying (H1)–(H3) and (H4′) is given by

$$H(x,p) = F(x,p) - 1$$

where $F \in C(\overline{\Omega} \times \mathbb{R}^d)$ is assumed to be positively homogeneous convex in p with the bounds

$$0 < F_* \leqslant F(x,p) \leqslant F^*, \quad x \in \overline{\Omega}, \ \|p\| = 1. \tag{7}$$

Duality theory of nonnegative positively homogeneous convex function (*gauges*) (Rockafellar 1970, Sect. 15) teaches that the support function ρ of the zero-level set of H is the *polar* of F, that is,

$$\rho(x,q) = \max_{p \neq 0} \frac{\langle p,q \rangle}{F(x,p)}, \quad F(x,p) = \max_{q \neq 0} \frac{\langle p,q \rangle}{\rho(x,q)}.$$

Hence, for the (particular) Hamilton–Jacobi–Bellman equation [Sethian and Vladimirsky 2003; Eq. (22)] with

$$H(x,p) = \max_{\|q\|=1} \langle p, -q \rangle f(x,q) - 1, \tag{8}$$

where the continuous speed function f with bounds $0 < f_* \leqslant f(x,q) \leqslant f^*$, $x \in \overline{\Omega}$, $\|q\| = 1$, has convex profiles $\{f(x, -q/\|q\|) \cdot q : \|q\| \leqslant 1\}$, we immediately read off that

$$\rho(x,q) = \frac{\|q\|}{f(x, -q/\|q\|)}, \quad x \in \overline{\Omega},\ q \in \mathbb{R}^d. \tag{9}$$

[1] Which essentially covers the general case as we will see in Footnote 2 in Sect. 6.

4 The finite-element discretization

4.1 Linear finite elements

Let us shortly recall the notion of linear finite-elements. For a sequence $h \to 0$ we consider a family Σ_h of shape-regular simplicial triangulations of (the now polytopal domain) $\Omega \subset \mathbb{R}^d$. We denote the diameter of a (closed) simplex $\sigma \in \Sigma_h$ by $h_1(\sigma)$ and the minimal height of a vertex in σ by $h_0(\sigma)$. We assume

$$h = \max_{\sigma \in \Sigma_h} h_1(\sigma)$$

and measure the shape-regularity by a uniform bound

$$1 \leqslant \frac{h_1(\sigma)}{h_0(\sigma)} \leqslant \theta, \quad \sigma \in \Sigma_h,\ h \to 0,$$

where we call θ the *regularity constant* of the family of triangulations.

The space of linear finite elements on Σ_h, that is, continuous functions that are affine if restricted to a simplex $\sigma \in \Sigma_h$, is denoted by V_h and

$$I_h : C(\overline{\Omega}) \to V_h$$

is the corresponding nodal interpolation operator. We endow V_h with the maximum norm, that is, convergence in V_h is the uniform convergence of the finite-element functions.

The set of nodal points (vertices) of the triangulation Σ_h that belong to $\overline{\Omega}$, Ω, $\partial\Omega$ are denoted by $\overline{\Omega}_h$, Ω_h, $\partial\Omega_h$, respectively. Note that a finite-element function $u_h \in V_h$ is uniquely determined by its nodal values, that is, the values $u_h(x_h)$ for all $x_h \in \overline{\Omega}_h$.

For an interior nodal point $x_h \in \Omega_h$ we consider the simplicial neighborhood $\omega_h(x_h)$, that is, the interior of the union of all simplices in Σ_h that have x_h as a vertex (see Fig. 1).

4.2 The idea

The finite-element discretization which we propose is motivated by the idea of *local solutions*:

> At $x_h \in \Omega_h$ the finite-element solution $u_h \in V_h$ takes the value $u_h^*(x_h)$ of the *exact* viscosity solution $u_h^* \in C^{0,1}(\overline{\omega_h(x_h)})$ that solves a simplified Hamilton–Jacobi equation on $\omega_h(x_h)$ subject to the boundary conditions $u_h^*|_{\partial\omega_h(x_h)} = u_h|_{\partial\omega_h(x_h)}$.

A good candidate for such a simplification of the Hamilton–Jacobi equation (4) is obtained by freezing locally the dependence of H on its first variable. This way $u_h^* \in C^{0,1}(\overline{\omega_h(x_h)})$ is obtained as the viscosity solution of the *local Dirichlet problem*

$$H(x_h, Du_h^*(x)) = 0 \ \text{ on } \omega(x_h), \quad u_h^*|_{\partial\omega_h(x_h)} = u_h|_{\partial\omega_h(x_h)}. \tag{10}$$

This simplification is particularly suitable, because for $[y,z] \subset \overline{\omega_h(x_h)}$ the optical distance $\delta_{x_h}(y,z)$ of the local equation (10) is, by Lemma 4,

$$\delta_{x_h}(y,z) = \rho(x_h, y - z),$$

where $\rho(x_h,\cdot)$ is the support function of the zero-level set of the convex function $H(x_h,\cdot)$ as defined in (6). Theorem 1 tells us that *if* the local Dirichlet problem (10) is solvable, then the value $u_h^*(x_h)$ is given by the *Hopf–Lax formula*

$$u_h^*(x_h) = \min_{y\in\partial\omega_h(x_h)} \Big(u_h(y) + \rho(x_h, x_h - y)\Big),$$

that is, by a simple *local variational principle*. We note that, under the assumptions (H1)–(H4), the Hopf–Lax formula is well-defined *independently* of the compatibility of the boundary data of the local Dirichlet problem (10). Anyway, we base the finite-element discretization on this formula and the convergence will be proved later without the interpretation of u_h^* as a local solution.

4.3 The discretization

We define a function $\Lambda_h : V_h \to V_h$, called the *Hopf–Lax update* function, by

$$(\Lambda_h u_h)(x_h) = \begin{cases} \min\limits_{y\in\partial\omega_h(x_h)} \Big(u_h(y) + \rho(x_h, x_h - y)\Big), & x_h \in \Omega_h, \\ u_h(x_h), & x_h \in \partial\Omega_h. \end{cases}$$

The finite-element solution $u_h \in V_h$ that discretizes the Dirichlet problem (1) is now *implicitly* defined by the fixed-point equation

$$u_h = \Lambda_h u_h, \quad u_h|_{\partial\Omega_h} = g|_{\partial\Omega_h}. \tag{11}$$

As in the continuous case, we call $u_h \in V_h$ a finite-element subsolution (supersolution) if $u_h \leqslant \Lambda_h u_h$ ($u_h \geqslant \Lambda_h u_h$).

We remark that the evaluation of $(\Lambda_h u_h)(x_h)$ at an interior nodal point $x_h \in \Omega_h$ can be calculated by a finite collection of $(d-1)$-dimensional convex optimization problems. This follows from the representation (see Fig. 1)

$$(\Lambda_h u_h)(x_h) = \min_{\sigma\in\Sigma_h : x_h\in\sigma} \min \Big\{ u_h(y) + \rho(x_h, x_h - y) : \\ y \in \text{the } (d-1)\text{-dimensional face of } \\ \sigma \text{ opposite to } x_h \Big\} \tag{12}$$

and the observation that u_h is affine, and hence $u_h + \rho(x_h, x_h - \cdot)$ convex, on σ. In Sect. 7 we will study an important class of examples for which these $(d-1)$-dimensional convex optimization problems allow for a particularly simple solution. In general, however, one would have to use suitable iterative numerical methods to solve them.

Remark For the particular Hamilton–Jacobi–Bellman equation (8) the finite-element discretization (11) is equivalent to various grid-based methods that are obtained from linear interpolation of the grid values and a direct local application of Bellman's dynamic programming principle. See, e.g., [Tsitsiklis 1995, Eq. (2.3)] and [Sethian and Vladimirsky 2003, Eq. (25)] as well as the references given therein.

5 Existence and uniqueness of the finite-element solution

The existence of a finite-element solution as implicitly defined by (11) is based on two simple properties of the Hopf–Lax update function Λ_h.

Lemma 5 *Assume (H1)–(H4). Let $u_h, v_h \in V_h$.*

1. *Λ_h is monotone, that is, $u_h \leqslant v_h$ implies $\Lambda_h u_h \leqslant \Lambda_h v_h$.*
2. *Λ_h is nonexpanding, that is, $\|\Lambda_h u_h - \Lambda_h v_h\|_\infty \leqslant \|u_h - v_h\|_\infty$.*

Proof The first property is an immediate consequence of the definition of Λ_h. To prove the second, let the maximum be attained at a nodal point $x_h \in \overline{\Omega}_h$, without loss of generality $\|\Lambda_h u_h - \Lambda_h v_h\|_\infty = (\Lambda_h u_h)(x_h) - (\Lambda_h v_h)(x_h)$. If $x_h \in \partial\Omega_h$ there is nothing to show; so we can assume $x_h \in \Omega_h$. Let $y_* \in \partial\omega_h(x_h)$ be such that

$$(\Lambda_h v_h)(x_h) = v_h(y_*) + \rho(x_h, x_h - y_*). \tag{13}$$

Hence

$$(\Lambda_h u_h)(x_h) - (\Lambda_h v_h)(x_h) \leqslant \Big(u_h(y_*) + \rho(x_h, x_h - y_*)\Big) - \Big(v_h(y_*) + \rho(x_h, x_h - y_*)\Big) \leqslant \|u_h - v_h\|_\infty\,,$$

which proves the assertion. □

Theorem 6 *Assume (H1)–(H4) and* $g : \partial\Omega \to \mathbb{R}$. *Then the finite-element discretization (11) has a solution* $u_h \in V_h$. *If* $u_h^0 \in V_h$ *is such that* $u_h^0|_{\partial\Omega_h} = g|_{\partial\Omega_h}$ *and* $\Lambda_h u_h^0 \geqslant u_h^0$, *then the fixed point iteration*

$$u_h^{n+1} = \Lambda_h u_h^n, \quad n = 0, 1, 2, \ldots,$$

converges monotonously to a solution of (11).

Proof An initial iterate $u_h^0 \in V_h$ with $\Lambda_h u_h^0 \geqslant u_h^0$ is given by

$$u_h^0|_{\partial\Omega_h} = g|_{\partial\Omega_h}, \quad u_h|_{\Omega_h} = \min_{x \in \partial\Omega_h} g(x).$$

Inductively the monotonicity of Λ_h implies $u_h^{n+1} = \Lambda_h u_h^n \geqslant u_h^n$. Hence, the monotone convergence of the sequence follows if we establish a uniform bound on the iterates. Since such a bound is trivial for the boundary we consider for a given $x_h \in \Omega_h$ a shortest path $x_h = x_h^0, \ldots, x_h^m$ of nodal points that connects x_h along edges of the triangulation with the boundary: $x_h^m \in \partial\Omega_h$. There is a bound L on the length of such a path which depends on the triangulation but not on x_h. With ρ^* as defined in Lemma 3 we get

$$u_h^n(x_h) \leqslant \max_{y \in \partial\Omega_h} g(y) + \sum_{i=0}^{m-1} \rho(x_h^i, x_h^i - x_h^{i+1}) \leqslant \max_{y \in \partial\Omega_h} g(y) + \rho^* L. \tag{14}$$

Thus, $u_h^n \to u_h \in V_h$ for some $u_h \in V_h$, which by continuity must be a fixed point of Λ_h. □

Like in the continuous case, uniqueness of the finite-element solution requires the sharper condition (H4′) and is a simple corollary of the following *discrete comparison principle*. Thus, the finite-element discretization is a *monotone scheme*.

Theorem 7 *Assume (H1)–(H3) and (H4′). Let* $u_h, v_h \in V_h$ *be finite-element sub- and supersolutions, respectively. If* $u_h \leqslant v_h$ *on* $\partial\Omega_h$ *then* $u_h \leqslant v_h$ *on* $\overline{\Omega}$.

Proof Let be $\Delta_h = u_h - v_h \in V_h$. Note that the maximum of Δ_h will be attained in a nodal point. We will show that the existence of $x_h \in \Omega_h$ with $\Delta_h(x_h) = \max_{x \in \overline{\Omega}} \Delta_h(x) = \delta > 0$ yields a contradiction. To this end we choose such a maximizing x_h with minimal value of $v_h(x_h)$. With $y_* \in \partial\omega_h(x_h)$ as in (13) we get

$$\delta = u_h(x_h) - v_h(x_h) \leqslant (\Lambda_h u_h)(x_h) - (\Lambda_h v_h)(x_h) \leqslant u_h(y_*) - v_h(y_*).$$

On the boundary of the face that contains y_* in its relative interior there is, by the maximality of δ, a point $x_h^* \in \Omega_h$ such that $\Delta_h(x_h^*) = \delta$ and $v_h(x_h^*) \leqslant v_h(y_*)$. By Lemma 3 we have $\rho(x_h, x_h - y_*) > 0$ and obtain

$$v_h(x_h^*) \leqslant v_h(y_*) = (\Lambda_h v_h)(x_h) - \rho(x_h, x_h - y_*) < (\Lambda_h v_h)(x_h) \leqslant v_h(x_h)$$

in contradiction to the minimality of $v_h(x_h)$. □

In the discrete case, up to now, we did not impose a compatibility condition on the boundary data such as (H5). This will change in the discussion of a third important property of the finite-element solutions needed for convergence, that is, uniform Lipschitz continuity. Based on the constant $\rho_* \geqslant 0$ of Lemma 4 and the regularity constant $\theta \geqslant 1$ of the family of triangulation we consider the condition

$$\text{(H5}'\text{)} \quad g(x) - g(y) \leqslant \frac{\rho_*}{\theta}\,\|x - y\| \quad \text{for all } x, y \in \partial\Omega.$$

By Lemma 4 this condition is actually stronger than (H5). Note that the *homogeneous* Dirichlet condition $g = 0$ always satisfies (H5′).

Theorem 8 *Assume (H1)–(H3), (H4′) and (H5′). The unique finite-element solution* $u_h \in V_h$ *of (11) satisfies the uniform Lipschitz condition*

$$|u_h(x) - u_h(y)| \leqslant c_\Omega\, \theta d \cdot \rho^* \cdot \|x - y\|\,, \quad x, y \in \overline{\Omega},$$

and the uniform bound

$$\|u_h\|_\infty \leqslant \max_{x \in \partial\Omega} |g(x)| + c_\Omega\, \theta d \cdot \rho^* \cdot \operatorname{diam}(\Omega).$$

Here, θ *denotes the regularity constant of the family of triangulations,* ρ^* *is the constant defined in Lemma 3 and* $c_\Omega > 0$ *is a constant depending only on* Ω. *If* Ω *is convex, we can choose* $c_\Omega = 1$.

Proof The uniform bound on $\|u_h\|_\infty$ is a simple consequence of the Lipschitz condition. The proof of the Lipschitz condition proceeds in three steps, imposing less and less restrictions on the possible choices of $x, y \in \overline{\Omega}$.

Step 1 For *neighboring* nodal points $x_h, y_h \in \overline{\Omega}_h$ we prove

$$|u_h(x_h) - u_h(y_h)| \leqslant \rho^* \cdot \|x_h - y_h\|\,.$$

Since $\rho^* \geqslant \rho_* \geqslant \rho_*/\theta$ this is, by (H5′), obviously true for $x_h, y_h \in \partial\Omega_h$. If $x_h \in \Omega_h$ we have $y_h \in \partial\omega_h(x_h)$ and hence

$$\begin{aligned} u_h(x_h) &= (\Lambda_h u_h)(x_h) \leqslant u_h(y_h) + \rho(x_h, x_h - y_h) \\ &\leqslant u_h(y_h) + \rho^* \|x_h - y_h\| . \end{aligned}$$

If $y_h \in \Omega_h$ we can change the roles of x_h and y_h and the Lipschitz bound follows.

Assume on the other hand that $y_h \in \partial\Omega_h$. There is a minimizing $y_* \in \partial\omega_h(x_h)$ such that

$$u_h(x_h) = (\Lambda_h u_h)(x_h) = u_h(y_*) + \rho(x_h, x_h - y_*) > u_h(y_*),$$

where the last inequality follows from Lemma 3. The boundary of the face that contains y_* in its relative interior has a point $x_h^1 \in \overline{\Omega}_h$ with $u_h(x_h^1) \leqslant u_h(y_*) < u_h(x_h)$. By the definition of ρ_* and θ we obtain

$$\rho(x_h, x_h - y_*) \geqslant \rho_* \|x_h - y_*\| \geqslant \frac{\rho_*}{\theta} \|x_h - x_h^1\|.$$

Continuing this construction we obtain a sequence $x_h = x_h^0, x_h^1, \ldots, x_h^m$ of nodal points with strictly decreasing u_h-values that necessarily reaches the boundary at some index m: $x_h^m \in \partial\Omega_h$. Thus, by construction and (H5′),

$$\begin{aligned} u_h(x_h) &\geqslant g(x_h^m) + \frac{\rho_*}{\theta} \sum_{i=0}^{m-1} \|x_h^i - x_h^{i+1}\| \\ &\geqslant g(y_h) + \frac{\rho_*}{\theta} \left(\sum_{i=0}^{m-1} \|x_h^i - x_h^{i+1}\| - \|x_h^m - y_h\| \right) \\ &\geqslant u_h(y_h) - \frac{\rho_*}{\theta} \|x_h - y_h\| \geqslant u_h(y_h) - \rho^* \|x_h - y_h\| , \end{aligned}$$

which concludes the proof of Step 1.

Step 2 Let $\sigma \in \Sigma_h$ be a simplex of the triangulation. For $x, y \in \sigma$ we prove that

$$|u_h(x) - u_h(y)| \leqslant \theta d \cdot \rho^* \cdot \|x - y\| .$$

By an affine transformation $\hat{x} \mapsto B\hat{x} + b$ we map the standard d-dimensional simplex

$$\hat{\sigma} = \left\{\hat{x} \in \mathbb{R}^d_{\geqslant 0} : \hat{x}_1 + \cdots + \hat{x}_d \leqslant 1\right\}$$

onto σ. The pullback of $u_h|_\sigma$ under the transformation will be denoted $\hat{u}$. By Step 1 we can estimate the length of the (constant) gradient of $u_h|_\sigma$ by

$$\|Du_h|_\sigma\| \leqslant \|B^{-1}\| \, \|D\hat{u}\| \leqslant \|B^{-1}\| \sqrt{d}\, \rho^* \cdot h_1(\sigma).$$

Now, $\|B^{-1}\|$ is the largest ratio of the length of a segment in $\hat{\sigma}$ to the length of its image in σ. Without loss of generality such a segment can be assumed to join a vertex with the opposite boundary face. Thus $\|B^{-1}\| \leqslant \sqrt{d}/h_0(\sigma)$ and, by the shape-regularity assumption, that is, $h_1(\sigma)/h_0(\sigma) \leqslant \theta$, we get

$$\|Du_h|_\sigma\| \leqslant \theta d \cdot \rho^*$$

and hence the assertion of Step 2.

Step 3 For $x, y \in \overline{\Omega}$ there is a Lipschitz path $\gamma \in C^{0,1}([0,1], \overline{\Omega})$ joining x and y such that (see Alt 1999, p. 304)

$$\|\gamma'\|_\infty \leqslant c_\Omega \|x - y\|.$$

For convex Ω the path γ can be chosen as the segment joining x and y, which yields $c_\Omega = 1$. Now, let $0 = t_0 < t_1 < \cdots < t_m = 1$ be a subdivision of $[0,1]$ such that $\gamma(t_{i-1})$ and $\gamma(t_i)$ are elements of a common simplex. By Step 2 we obtain

$$\begin{aligned} |u_h(x) - u_h(y)| &\leqslant \sum_{i=0}^{m-1} |u_h(\gamma(t_i)) - u_h(\gamma(t_{i+1}))| \leqslant \theta d \cdot \rho^* \\ &\quad \times \sum_{i=0}^{m-1} \|\gamma(t_i) - \gamma(t_{i+1})\| \\ &\leqslant c_\Omega \theta d \cdot \rho^* \cdot \|x - y\| \\ &\quad \times \sum_{i=0}^{m-1} |t_i - t_{i+1}| = c_\Omega \theta d \cdot \rho^* \cdot \|x - y\|, \end{aligned}$$

which concludes the proof of the asserted Lipschitz bound. □

6 Convergence of the finite-element discretization

The argument will be simplified if we consider a modified Hamiltonian $\tilde{H}$ for which the corresponding Hamilton–Jacobi equation possesses the same viscosity solutions as the original one.

Lemma 9 *Assume (H1)–(H4). Let $x \in \bar{\Omega}$ and $p \in \mathbb{R}^d$. For the modified Hamiltonian*

$$\tilde{H}(x,p) = \max_{\|q\|=1} (\langle p, q\rangle - \rho(x,q))$$

we get that $\tilde{H}(x,p) \leqslant 0$ ($\tilde{H}(x,p) \geqslant 0$) implies $H(x,p) \leqslant 0$ ($H(x,p) \geqslant 0$).

Proof First assume $H(x,p) > 0$. There is a hyperplane that separates p strongly from the compact and convex level set $\{\tilde{p} : H(x, \tilde{p}) \leqslant 0\}$ (see Rockafellar 1970, Cor. 11.4.2). That is, there is a vector $q \in \mathbb{R}^d$, $\|q\| = 1$, with $\rho(x,q) < \langle p, q\rangle$. Hence

$$\tilde{H}(x,p) \geqslant \langle p, q\rangle - \rho(x,q) > 0.$$

Now assume $H(x,p) < 0$. There is $\epsilon > 0$ such that $H(x, p + \delta p) < 0$ for $\|\delta p\| \leqslant \epsilon$. Hence

$$\langle p, q\rangle - \rho(x,q) \leqslant \langle p, q\rangle - \max_{\|\delta p\| \leqslant \epsilon} \langle p + \delta p, q\rangle = -\epsilon \|q\| ,$$

Taking the supremum over all q with $\|q\| = 1$ yields $\tilde{H}(x,p) \leqslant -\epsilon < 0$. □

In particular, each viscosity subsolution (supersolution) of the thus modified Hamilton–Jacobi equation $\tilde{H}(x, Du(x)) = 0$, $x \in \Omega$, is also a viscosity subsolution (supersolution) of the original one $H(x, Du(x)) = 0$, $x \in \Omega$.[2]

Loosely speaking, in the framework of viscosity solutions the notion of *consistency* of a discretization means that a smooth function is already a subsolution (supersolution) of the differential equation if it is a subsolution (supersolution) of the discrete scheme. The precise statement is given in the next theorem.

Theorem 10 *Assume (H1)–(H3) and (H4′). Let $v \in C_0^\infty(\Omega)$, $x \in \Omega$, and $x_h \in \Omega_h$ be a sequence of nodal points that converges to x as $h \to 0$. Then*

$$v(x_h) \leqslant (\Lambda_h I_h v)(x_h) \quad \text{for all } h \;\Rightarrow\; H(x, Dv(x)) \leqslant 0,$$
$$v(x_h) \geqslant (\Lambda_h I_h v)(x_h) \quad \text{for all } h \;\Rightarrow\; H(x, Dv(x)) \geqslant 0,$$

where $I_h : C(\overline{\Omega}) \to V_h$ denotes the nodal interpolation operator.

Proof Since v is smooth we can approximate the directional derivatives of v in x_h by first order differences as follows

$$\frac{v_h(x_h) - (I_h v)(y)}{\|x_h - y\|} = \left\langle Dv(x_h), \frac{x_h - y}{\|x_h - y\|} \right\rangle + O(h), \qquad y \in \partial\omega_h(x_h). \tag{15}$$

Now, let $v(x_h) \leqslant (\Lambda_h I_h v)(x_h)$ for all h of the sequence, that is,

$$v(x_h) - (I_h v)(y) - \rho(x_h, x_h - y) \leqslant 0, \quad y \in \partial\omega_h(x_h).$$

After division by $\|x_h - y\|$ we get, by (15), a constant $c > 0$ such that

$$\langle Dv(x_h), q\rangle - \rho(x_h, q) \leqslant ch, \quad \|q\| = 1.$$

If we pass to the limit $h \to 0$ (note the continuity of ρ at $x \in \Omega$ as stated in Lemma 3) and take thereafter the maximum over all $\|q\| = 1$, we obtain

$$\tilde{H}(x, Dv(x)) = \max_{\|q\|=1} (\langle Dv(x), q\rangle - \rho(x, q)) \leqslant 0.$$

From Lemma 9 we infer the assertion $H(x, Dv(x)) \leqslant 0$.

On the other hand, let $v(x_h) \geqslant (\Lambda_h I_h v)(x_h)$ for all h of the sequence, that is,

$$v(x_h) - (I_h v)(y_h) - \rho(x_h, x_h - y_h) \geqslant 0$$

for some $y_h \in \partial\omega_h(x_h)$. After division by $\|x_h - y_h\|$ we get, by (15), a constant $c > 0$ such that

$$\langle Dv(x_h), q_h\rangle - \rho(x_h, q_h) \geqslant -ch, \quad q_h = (x_h - y_h)/\|x_h - y_h\|.$$

By compactness, we can assume that $q_h \to q_*$ with $\|q_*\| = 1$. Passing to the limit $h \to 0$ we thus obtain

$$\tilde{H}(x, Dv(x)) \geqslant \langle Dv(x), q_*\rangle - \rho(x, q_*) \geqslant 0,$$

from which we infer the assertion $H(x, Dv(x)) \geqslant 0$ by Lemma 9. □

Now we have all the tools in hand to prove the *convergence* of the finite-element discretization.

Theorem 11 *Assume (H1)–(H3), (H4′), and (H5′). Then, as $h \to 0$, the sequence of unique finite-element solutions $u_h \in V_h$ defined by*

$$u_h = \Lambda_h u_h, \quad u_h|_{\partial\Omega_h} = g|_{\partial\Omega_h},$$

converges uniformly to the unique viscosity solution u of the Dirichlet problem

$$H(x, Du(x)) = 0, \quad u|_{\partial\Omega} = g.$$

Proof Theorems 6 and 7 show the existence and uniqueness of the finite-element solutions $u_h \in V_h$. Theorem 8 shows that $u_h \in V_h$ is a uniform bounded sequence of uniform Lipschitz continuous functions. By the theorem of Arzelà–Ascoli there is a subsequence $(u_{h'})$ that converges uniformly to a function $u \in C^{0,1}(\overline{\Omega})$. Because of (H5′) and $u_h|_{\partial\Omega_h} = g|_{\partial\Omega_h}$, this limit satisfies the boundary condition $u|_{\partial\Omega} = g$.

To show that u is a viscosity *subsolution* of $H(x, Du(x)) = 0$ let $v \in C_0^\infty(\Omega)$ and $x_0 \in \Omega$ such that $u - v$ attains a local maximum in x_0. By adding a quadratic parabola to v if necessary, we may assume that it is in fact a strict local maximum (see Evans 1998, p. 542). Extracting a further subsequence of h' if necessary, there is, by uniform convergence and the monotonicity of the nodal interpolation operator $I_h : C(\overline{\Omega}) \to V_h$, a sequence of nodal points $x_{h'} \in \Omega_{h'}$ such that $x_{h'} \to x_0$ and (see the argument given by Evans 1998)

$$(u_{h'} - v)(x_{h'}) \geqslant (u_{h'} - I_{h'} v)(y), \quad y \in \partial\omega_{h'}(x_{h'}).$$

Now let $y_* \in \partial\omega_{h'}(x_{h'})$ be a minimizing argument such that

$$(\Lambda_{h'} I_{h'} v)(x_{h'}) = (I_{h'} v)(y_*) + \rho(x_{h'}, x_{h'} - y_*).$$

Then it holds that

$$\begin{aligned}(u_{h'} - v)(x_{h'}) &\geqslant u_{h'}(y_*) + \rho(x_{h'}, x_{h'} - y_*) - (I_{h'} v)(y_*) \\ &\quad - \rho(x_{h'}, x_{h'} - y_*) \\ &\geqslant (\Lambda_{h'} u_{h'} - \Lambda_{h'} I_{h'} v)(x_{h'}) \\ &= (u_{h'} - \Lambda_{h'} I_{h'} v_{h'})(x_{h'})\end{aligned}$$

[2] Under the additional assumption (H4′) the same holds true, since then $\rho(x, q) > 0$ for $x \in \Omega$, if we consider the modified Hamilton–Jacobi equation with the Hamiltonian

$$\tilde{H}(x, p) = \max_{\|q\|=1} \frac{\langle p, q\rangle}{\rho(x, q)} - 1.$$

Hence, we see that the example at the end of Sect. 2 in fact covers the general case.

and thus

$$v(x_{h'}) \leqslant (\Lambda_{h'} I_{h'} v)(x_{h'}).$$

The consistency of the discretization, stated in Theorem 10, yields that

$$H(x_0, Dv(x_0)) \leqslant 0,$$

which concludes the proof that u is a viscosity subsolution.

In the same way we prove that u is a viscosity *supersolution* of $H(x, Du(x)) = 0$. Therefore, u is a viscosity solution, which, by the comparison principle (Theorem 2), is actually *unique*. Hence, there is exactly one limit point of the sequence u_h, which thus has to converge uniformly to the just established viscosity solution u. □

Remark Note that the only use that we have made so far of the existence Theorem 1 was to motivate the local variational principle for the finite-element discretization. In fact, our proof of the convergence result shows the existence of a viscosity solution *en route* — under the somewhat more restrictive compatibility conditions (H4′) and (H5′), however.

7 The Hopf–Lax update for generalized eikonal equations in 2D

Let $\Omega \subset \mathbb{R}^2$ be a polygonal Lipschitz domain and $M : \overline{\Omega} \to \mathbb{R}^{2\times 2}$ be a continuous mapping into the *symmetric positive definite* 2×2-matrices. We denote the corresponding inner product by $\langle p, q\rangle_{M(x)} = \langle M(x)p, q\rangle$, its subordinate norm by $\|p\|_{M(x)} = \langle p, p\rangle_{M(x)}^{1/2}$.

Now, we consider the Dirichlet problem for the *generalized eikonal equation*,

$$\|Du\|_{M(x)} = 1 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = g.$$

Its Hamiltonian $H(x, p) = \|p\|_{M(x)} - 1$ satisfies the assumptions (H1)–(H3) and (H4′). The support function of the zero-level set is simply given by the norm that is dual to $\|\cdot\|_{M(x)}$, namely,

$$\rho(x, q) = \max_{H(x,p)=0} \langle p, q\rangle = \max_{\|p\|_{M(x)}=1} \langle p, q\rangle = \|q\|_{M(x)^{-1}}.$$

The Hopf–Lax update function becomes

$$(\Lambda_h u_h)(x_h) = \min_{y\in\partial\omega_h(x_h)} \Big(u_h(y) + \|x_h - y\|_{M(x_h)^{-1}}\Big), \qquad x_h \in \Omega_h,\ u_h \in V_h.$$

There is a simple procedure to evaluate $(\Lambda_h u_h)(x_h)$ at $x_h \in \Omega_h$. To this end let $\sigma_1, \ldots, \sigma_m \in \Sigma_h$ be the triangles that have x_h as a vertex and J_i the (closed) edge of σ_i opposite to x_h. Then, as in (12),

$$(\Lambda_h u_h)(x_h) = \min_{1\leqslant i\leqslant m} u_i \quad \text{with}$$

$$u_i = \min_{y\in J_i} \Big(u_h(y) + \|x_h - y\|_{M(x_h)^{-1}}\Big).$$

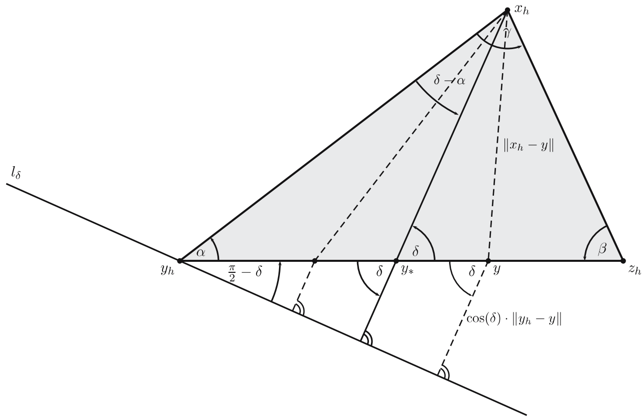

Fig. 2 Geometry of the minimization of $\cos(\delta)\|y - y_h\| + \|x_h - y\|$ for $y \in [y_h, z_h]$. Note that for $\delta > \pi/2$ the segment through y perpendicular to l_δ has *negative* length $\cos(\delta) \cdot \|y - y_h\|$

Let us take one of the triangles, σ_i, (see Fig. 1) and call its vertices x_h, y_h, z_h, hence $J_i = [y_h, z_h]$. In the case of the classic eikonal equation, that is, $M(x) \equiv I$, the update u_i can be determined from an elementary geometric argument.

Lemma 12 *Let $\sigma \in \Sigma_h$ be the triangle with the vertices x_h, y_h, z_h and $u_h \in V_h$. Denote the angles at y_h, z_h by α, β, respectively. Defining*

$$\Delta = \frac{u_h(z_h) - u_h(y_h)}{\|z_h - y_h\|}$$

and $\cos(\delta) = \Delta$ if $|\Delta| \leqslant 1$, we obtain

$$u_i = \min_{y\in[y_h,z_h]} \Big(u_h(y) + \|x_h - y\|\Big) = u_h(y_h) + \min_{y\in[y_h,z_h]} \Big(\Delta\cdot\|y - y_h\| + \|x_h - y\|\Big)$$

$$= \begin{cases} u_h(y_h) + \|x_h - y_h\|, & \cos(\alpha) \leqslant \Delta, \\ u_h(y_h) + \cos(\delta - \alpha)\cdot\|x_h - y_h\|, & \alpha \leqslant \delta \leqslant \pi - \beta, \\ u_h(z_h) + \|x_h - z_h\|, & \Delta \leqslant \cos(\pi - \beta). \end{cases}$$

Proof For $|\Delta| \geqslant 1$ the assertion follows from a direct application of the triangle inequality; e.g., for $\Delta \geqslant 1$,

$$\Delta\cdot\|y - y_h\| + \|x_h - y\| \geqslant \|y - y_h\| + \|x_h - y\| \geqslant \|x_h - y_h\|.$$

Now, let $|\Delta| < 1$ so that $\cos(\delta) = \Delta$ defines a $\delta \in (0, \pi)$. A look at Fig. 2 shows that

$$\cos(\delta)\cdot\|y - y_h\| + \|x_h - y\| \tag{16}$$

attains its minimum at the unique intersection y_* of two straight lines: the first line running through y_h and z_h, the second line running through x_h perpendicular to l_δ. Here, l_δ is the straight line that encloses at y_h with $[y_h, z_h]$ the angle $\pi/2 - \delta$. We observe that the value of the minimum is simply $\cos(\delta - \alpha) \cdot \|x_h - y_h\|$. A further look at Fig. 2 teaches that $y_* \in [y_h, z_h]$ if and only if

$$0 \leqslant \delta - \alpha \leqslant \gamma = \pi - \alpha - \beta, \quad \text{that is, } \alpha \leqslant \delta \leqslant \pi - \beta.$$

If $\delta < \alpha$, or equivalently $\Delta > \cos(\alpha)$, y_* is to the left of y_h and the minimum of (16) in $[y_h, z_h]$ is attained at y_h. On the other hand, if $\delta > \pi - \beta$, or equivalently $\Delta < \cos(\pi - \beta)$, y_* is to the right of z_h and the minimum of (16) in $[y_h, z_h]$ is attained at z_h. □

For the general case we simple apply the triangular update formula of Lemma 12 to the image of the triangle σ_i under the linear transformation $M(x_h)^{-1/2}$. This way we immediately obtain the following update procedure, writing $\langle p, q\rangle_x = \langle p, q\rangle_{M(x)^{-1}}$, $\|p\|_x = \|p\|_{M(x)^{-1}}$, $c_\alpha = \cos(\alpha)$, and $c_\beta = \cos(\beta)$ for short (note that we used the addition formula to spell out $\cos(\alpha - \delta)$ for implementation purposes):

$$\Delta = \frac{u_h(z_h) - u_h(y_h)}{\|z_h - y_h\|_{x_h}};$$

$$c_\alpha = \frac{\langle x_h - y_h, z_h - y_h\rangle_{x_h}}{\|x_h - y_h\|_{x_h} \cdot \|z_h - y_h\|_{x_h}};$$

$$c_\beta = \frac{\langle x_h - z_h, y_h - z_h\rangle_{x_h}}{\|x_h - z_h\|_{x_h} \cdot \|y_h - z_h\|_{x_h}};$$

if $c_\alpha \leqslant \Delta$

$$u_i = u_h(y_h) + \|x_h - y_h\|_{x_h};$$

elseif $\Delta \leqslant -c_\beta$

$$u_i = u_h(z_h) + \|x_h - z_h\|_{x_h};$$

else

$$u_i = u_h(y_h) + \left(c_\alpha \Delta + \sqrt{(1 - c_\alpha^2)(1 - \Delta^2)}\right) \|x_h - y_h\|_{x_h};$$

Remark With different ideas on a discretization, exactly the same update formula has been obtained for the (classic) eikonal equation by Kimmel and Sethian (1998) (see also Sethian 1999, Sect. 10.3.1), who use for acute triangulations the methodology of Barth and Sethian (1998) to construct upwind schemes on unstructured meshes, and, independently, by the geophysicist Fomel (1997), who locally uses Fermat's principle of shortest travel times (which is closely related to our local use of the Hopf–Lax formula).

Sethian (1999, Sect. 10.1) shows further that this update formula generalizes the upwind finite-difference scheme on structured grids given by Rouy and Tourin (1992).

8 Solving the discrete system

8.1 A review of methods

Theorem 6 shows that the nonlinear discrete system (11) can be solved by the fixed-point iteration

$$u_h^{n+1} = \Lambda_h u_h^n, \quad n = 0, 1, 2, \ldots,$$

for a suitably chosen initial iterate u_h^0. Such a fixed-point iteration uses the updated values at a nodal point x_j only after all the updated values have been calculated. This corresponds to the classic *Jacobi–iteration* for linear systems of equations and lends itself to direct parallelization.

If we sequentially traverse the nodal points in a given order and modify the iteration to always use the most recently updated value, we obtain a nonlinear variant of the *Gauss–Seidel iteration*. Rouy and Tourin (1992) used such a nonlinear Gauss–Seidel iteration to solve a finite difference discretization of the eikonal equation $\|Du(x)\| = n(x)$ on a structured mesh.

For both iterative methods the complexity will typically scale as $O(N^{1+1/d})$, where N denotes the number of nodal points and d the space dimension. This is because the information about the solution, inherent initially to the boundary only, travels by next neighbor interaction at each run trough all nodal points. To spread that information to the whole computational domain about $O(N^{1/d})$ runs are necessary. However, even though this heuristic well explains the experimental observations, to our knowledge there is no rigorous proof of that in the literature.

Sethian (1996) and Tsitsiklis (1995) have shown independently that for eikonal equations on structured meshes the nonlinear equation can be solved *exactly* in a *single pass*, that is, by traversing the grid once using local operations only. This *fast marching method* was later generalized to triangular meshes by Kimmel and Sethian (1998). It relies on the *causality property*, namely that on an acute triangulation the value $u_h(x_h)$ depends only on the values in neighboring nodal points y_h that are lower, $u_h(y_h) \leqslant u_h(x_h)$. So the discrete solution can be computed starting from the neighborhood of the boundary moving further inwards the computational domain along increasing values of u_h. However, on non-acute triangulations additional effort is necessary to deal with the loss of this causality property (see Kimmel and Sethian 1998 for details). The complexity of this method is $O(N \log(N))$ where the logarithm comes from administering a priority queue of candidates for the next lowest value of u_h, such as a heap data structure.

For the particular Hamilton–Jacobi–Bellman equation (8) a single pass algorithm generalizing the fast marching method, called the *ordered upwind method* (OUM), was introduced by Sethian and Vladimirsky (2000) and is discussed in detail by Sethian and Vladimirsky (2003) . This method is not a fast solver for a given discretization, but the discretization is specifically designed for the needs of the fast solver. Like the Hopf–Lax update the update formulas of the OUM are based on local variational principles (Bellman's principle). However though, the OUM does not solve (11), since the update in x_h is not necessarily computed from the neighborhood $\omega_h(x_h)$ but from larger neighborhoods within the radius $h \cdot \nu$. Here $\nu = F^*/F_*$ denotes the *anisotropy coefficient* of the Hamilton–Jacobi–Bellman equation (see (7)). This quantity ν affects not only the complexity of the OUM, which is $O(\nu^{d-1} N \log(N))$, but also its accuracy.

8.2 Adaptive nonlinear Gauss–Seidel iteration

In this paper we propose an adaptive version of the nonlinear Gauss–Seidel iteration, which is a modelled after a similar relaxation method (Plaum and Rüde 1993) for the multilevel solution of elliptic boundary value problems. It turns out to be substantially faster than the standard Gauss–Seidel iteration, easy to implement and universal.

The adaptive Gauss–Seidel iteration differs from the standard one in two respects. First, like in the fast marching method, only those nodal points are updated that "have the information", that is, are neighbors of recently updated points. Second, the order of updates is not fixed but varies as the iteration proceeds. Thus, a queue denoted by $\mathcal{Q}$ is administered to provide the ordering of updates. However, other than in the fast-marching method where using the causality property requires to keep control of the point with minimal function value, the queue is now simply given the structure of a FIFO (first in first out) stack: the nodal point staying longest in the queue is updated next.

The algorithm is passed a user-defined tolerance tol and it ends up with an approximate finite-element solution $u_h \in V_h$ such that

$$\|u_h - \Lambda_h u_h\|_\infty \leqslant \text{tol}.$$

It is organized as follows:

(1) (Initialization) Let $u_h|_{\partial\Omega_h} = g|_{\partial\Omega_h}$, $u_h|_{\Omega_h} \equiv \infty$.[3] Let $\mathcal{Q}$ be the list of all points $x_h \in \Omega_h$ that are adjacent to some boundary point (in an arbitrary but fixed order).
(2) (Iteration) Remove the first point x_h from $\mathcal{Q}$ and compute the update value $u_{\text{new}} = (\Lambda_h u_h)(x_h)$.
(3) If $|u_{\text{new}} - u_h(x_h)| > \text{tol}$ then update $u_h(x_h) = u_{\text{new}}$ and append all not yet enqueued neighbors y_h of x_h to the queue $\mathcal{Q}$.
(4) If $\mathcal{Q} \neq \emptyset$, goto (2).

To prove the convergence of this method, we denote the initial finite-element function of step 1 by u_h^0. After the nth update has been performed in step 3 the actual finite-element function will be denoted by u_h^n.

Theorem 13 *The algorithm generates a sequence u_h^0, $u_h^1, \ldots$, that is monotonously decreasing. It terminates after finitely many steps with an approximate finite element solution $u_h \in V_h$, such that $\|u_h - \Lambda_h u_h\| \leqslant \text{tol}$.*

Proof The initialization $u_h|_{\Omega_h} \equiv \infty$ ensures that every point x_h is updated at least once, as the residual is ∞ when the first update value in x_h is computed. After the first update, $u_h(x_h)$ is assigned a finite value, since x_h has a neighbor in $\partial\Omega_h$ or a neighbor, for which a finitely valued update has already been computed. By induction on n we get that at each later update of a nodal point x_h, all neighbors of x_h that have been changed over the last update can only have been assigned a lower value of u_h. From the monotonicity of Λ_h we thus get the first assertion.

Since an update in step (3) only affects the residual in the neighboring points, which are are immediately enqueued, it holds that

$$\left\{x_h \in \Omega_h \ : \ u_h^n(x_h) < \infty \text{ and } |\Lambda_h u_h^n - u_h^n| > \text{tol}\right\} \subset \mathcal{Q}$$

for every $n \geqslant 0$. So if the algorithm terminates with $\mathcal{Q} = \emptyset$, the tolerance has been reached.

Otherwise, if the iteration does not terminate, then there is at least one nodal point x_h^* that appears infinitely often as the first element of the queue $\mathcal{Q}$ and gets updated at steps $n_j \to \infty$, $j \to \infty$. Hence, there must be $|u_h^{n_j}(x_h^*) - u_h^{n_j-1}(x_h^*)| > \text{tol}$ in contradiction to the convergence of $u_h^{n_j}(x_h^*)$ as $j \to \infty$ which is implied by the monotonicity and the trivial lower bound $u_h^n \geqslant \min_{x\in\partial\Omega} g(x)$. □

Though the run-time complexity of the adaptive Gauss–Seidel iteration behaves probably at worst as $O(N^{1+1/d})$ like in the standard Gauss–Seidel iteration, a lot of unnecessary updates are saved as we will see in the numerical experiments of the next section.

[3] In fact any value larger than the bound (14) will do. However, taking ∞ makes the argument more elegant and can correctly be implemented in IEEE arithmetic.

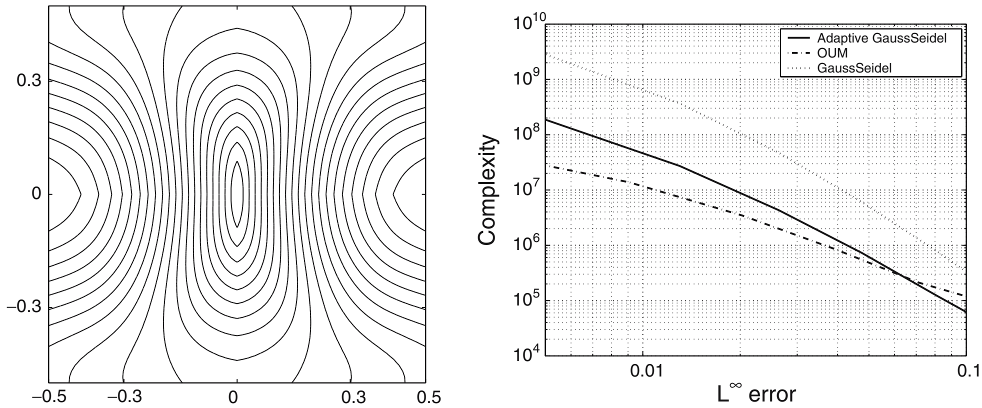

Fig. 3 *Left* Contour plot of the distance function over the parameter plane. *Right* Accuracy/complexity of adaptive Gauss–Seidel iteration in comparison to the Gauss–Seidel iteration and OUM

9 Numerical experiments

For the two following examples the solutions were computed on unstructured meshes with 23^2, 45^2, 91^2, 181^2, and 725^2 nodal points. A solution on a mesh with 1451^2 points served to estimate the discretization error. The iterative methods were used with an absolute tolerance $\mathrm{tol} = 10^{-8}$.

The first example concerns the distance map on the torus given by the immersion

$$f(x_1, x_2) = (\cos(2\pi x_1)(5 + 4\cos(2\pi x_2)),\\ \sin(2\pi x_1)(5 + 4\cos(2\pi x_2)), \sin(2\pi x_2))\,.$$

Using the Gram matrix $G(x) = Df(x)^T Df(x)$ and $\rho^2(x, q) = \langle q, G(x)q\rangle$ the distance between the points $f(x)$ and $f(y)$ on the manifold is given by the function $\delta(x, y)$ as defined in (2). With the results of Sect. 7 and (Lions 1982, Theorem 5.3(iv)) we obtain that $u(x) = \delta(x, 0)$ is the viscosity solution of the Dirichlet problem

$$\|Du\|_{G(x)^{-1}} = 1 \text{ on } \Omega \setminus \{0\}, \quad u(0) = 0,$$

where $\Omega = [-0.5, 0.5]^2$. The solution is shown to the left of Fig. 3 as a contour plot.

To the right of Fig. 3 we compare the accuracy and complexity of the adaptive Gauss-Seidel method with both the standard Gauss–Seidel iteration and the OUM.[4] Here, by *complexity* we mean the total number of updates calculated on a triangle by a formula such as the one at the end of Sect. 7. We observe that the adaptive Gauss–Seidel iteration is more than a factor of 10 faster than the standard Gauss–Seidel iteration but displays the same asymptotic rate of complexity. The OUM show, as theoretically expected, a better rate of complexity that, in this example, becomes significant even at larger tolerances.

The second example is taken from Sethian and Vladimirsky (2001) and shows the effect of a moderately large anisotropy coefficient $\nu = 19$, which affects the complexity of the OUM. We consider a simple min-time optimal control problem governed by the dynamical system

$$y'(t) = a(t) + b(y(t)), \quad y(0) = x.$$

The controls $a(\cdot)$ are taken from $\mathcal{A} = \{a : [0, \infty) \to S^1 \text{ measurable}\}$ and

$$b(y) = -0.9 \sin(4\pi y_1) \sin(4\pi y_2) \cdot \frac{y}{\|y\|}.$$

For $x \in \Omega = [-0.5, 0.5]^2$ and a control a we denote by $T_x(a)$ the minimal time that the trajectory $y(\cdot)$ takes to reach the origin.

Following (Bardi and Capuzzo-Dolcetta 1997, Theorem 2.6) the value function $u(x) = \inf_{a \in \mathcal{A}} T_x(a)$ is the viscosity solution of the Hamilton–Jacobi–Bellman equation

$$H(x, Du) = \max_{\|a\|=1} \langle a + b(x), -Du\rangle - 1 = 0, \quad u(0) = 0.$$

One figures out that $H(x, p) = \|p\| - \langle b(x), p\rangle - 1$; (H1), (H2), (H4′), and, as $\|b\| \leqslant 0.9$, the coercivity condition (H3) are fullfilled. A short calculation shows that

$$\rho(x, q) = \frac{\|q\|}{\left(1 - \|b(x)\|^2 + \langle b(x), q/\|q\|\rangle^2\right)^{1/2} - \langle b(x), q/\|q\|\rangle},$$

[4] We have coded the OUM from Sethian and Vladimirsky (2003) with a little completion that turned out to be necessary: Considered points have also to be updated, if they depend on an edge that drops out of the accepted front. If some point x_h gets accepted this may happen to any edge opposite to x_h in $\omega_h(x_h)$.

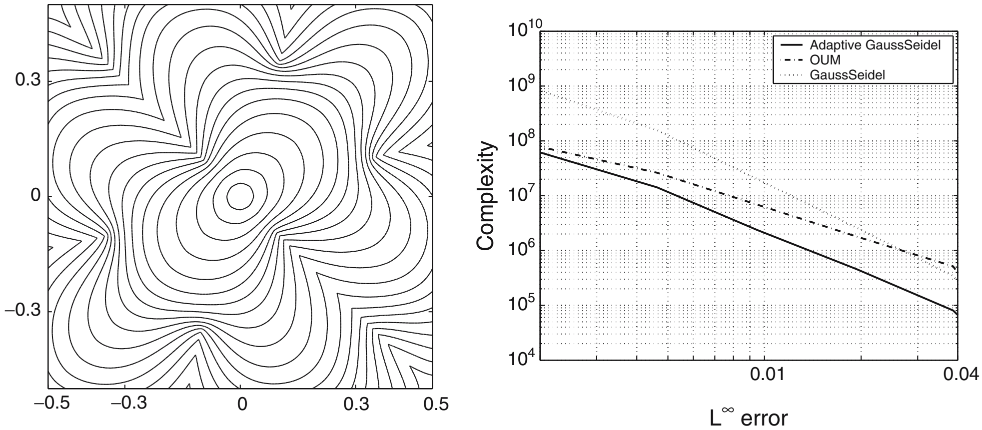

Fig. 4 *Left* Value function of some min-time optimal control problem. *Right* Complexity/accuracy of the methods in comparison

compare also (9) and (Sethian and Vladimirsky, Eq. (19)). The solution calculated on a 253×253 mesh can be found to the left of Figure 4. To the right of this figure the accuracy of the approximate finite-element solution is shown versus the complexity of the iteration. A comparison of the (adaptive) Gauss–Seidel iteration with the OUM is shown to the right of Fig. 4. Again we observe that the adaptive variant of the Gauss–Seidel iteration is by about a factor of ten more efficient than the standard one. This time, however, the quite sophisticated order upwind method behaves less favorable: because of the large anisotropy coefficient the break-even point at which the OUM becomes more efficient than the simple adaptive Gauss–Seidel iteration is at a mesh-size of more than $725^2 = 525{,}625$ nodal points. We expect this effect to become even more pronounced in 3D and higher, because of the increasingly better complexity rate of the Gauss–Seidel iteration.

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