On Target for Venus – Set Oriented Computation of Energy Efficient Low Thrust Trajectories

Michael Dellnitz[(1)] (dellnitz@math.upb.de), Oliver Junge[(1)] (junge@math.upb.de), Marcus Post[(1)] (mpost@math.upb.de) and Bianca Thiere[(1)] (thiere@math.upb.de)

[(1)] *Faculty of Computer Science, Electrical Engineering and Mathematics, University of Paderborn*
33095 Paderborn, Germany

Abstract. Recently new techniques for the design of energy efficient trajectories for space missions have been proposed that are based on the circular restricted three body problem as the underlying mathematical model. These techniques exploit the structure and geometry of certain invariant sets and associated invariant manifolds in phase space to systematically construct energy efficient flight paths.

In this paper we extend this model in order to account for a continuously applied control force on the spacecraft as realized by certain low thrust propulsion systems. We show how the techniques for the trajectory design can be suitably augmented and compute approximations to trajectories for a mission to Venus.

Keywords: dynamical systems, earth venus transfer, three body problem, low thrust trajectories, invariant manifolds, reachable sets, space mission design

1. Introduction

A new paradigm for the construction of energy efficient trajectories for spacecraft is currently emerging. It heavily bases on concepts and techniques from the theory and numerical treatment of dynamical systems. The basic strategy is the following: Instead of a two body problem, as in more classical approaches, one considers a restricted three body problem as the mathematical model for the motion of the spacecraft. This enables one to exploit the intricate structure and geometry of certain invariant sets and their stable and unstable manifolds – which are not present in two body problems – as candidate regions for energy efficient trajectories. For example, this approach has recently been used in the design of the trajectory for the *Genesis discovery mission*[1] (Lo et al., 2001).

Building on this basic concept, techniques have been proposed that synthesize partial orbits from different three body problems into a single one, yielding energy efficient trajectories with eventually very complicated itineraries (Koon et al., 2002; Koon et al., 2000b). In (Koon

[1] http://genesismission.jpl.nasa.gov

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et al., 2002), a *petit grand tour* between the moons of Jupiter has been constructed by this approach. The idea of the technique is as follows: One computes the intersection of parts of the stable resp. unstable manifold of two specific periodic orbits in the vicinity of two moons, respectively, with a suitably chosen surface. After a transformation of these two curves into a common coordinate system one identifies points on them that lie close to each other — ideally one searches for intersection points. Typically, however, these two curves will not intersect in the chosen surface, so a certain (impulsive) maneuver of the spacecraft will be necessary in order to transit from the part of the trajectory on the unstable manifold to the one on the stable manifold. In a final step this "coupled 3-body approximation" to a trajectory is used as an initial guess for standard local solvers using the full n-body dynamics of the solar system as the underlying model.

The approach of coupling 3-body problems as sketched above is tailored for spacecraft with impulsive thrust engines. Recently however, interest has grown in continuously thrusting engines that exert small forces on the spacecraft only. For these, the usual restricted three body problem is no adequate model, since one needs to incorporate the control forces.

In this paper we propose an extension of the coupled 3-body approach to the case of a continuously controlled spacecraft. Roughly speaking, the stable and unstable manifolds are replaced by certain (forward and backward) reachable sets in phase space. Using set oriented numerical tools we can efficiently compute the corresponding sets of intersection of these two reachable sets.

The paper is organized as follows: In Section 2 we briefly review the planar circular restricted three body problem that serves as a starting point for the discussion. Section 3 contains a sketch of the coupled 3-body technique. In Section 4 we present the augmented three body model that incorporates a continuously acting control acceleration. The description of the generalized coupling approach is given in Section 5, with comments on the implementation following in Section 6. In Section 7 we apply the procedure in order to compute a trajectory for a mission to Venus.

2. The Planar Circular Restricted Three Body Problem

As alluded to in the Introduction, typically the full n-body problem is too complicated for a detailed investigation of its dynamics. The classical patched conics approach breaks this model into several two-body problems whose solutions can easily be written down analytically. However, it turns out (McGehee, 1969; Koon et al., 2000a), that it is worthwhile to hazard the consequences of considering a more complicated model, the *(planar) circular restricted three body problem (PCR3BP)*.

Let us briefly recall the basics of this model — for a more detailed exposition see e.g. (Abraham and Marsden, 1978; Meyer and Hall, 1992; Szebehely, 1967). The PCR3BP models the motion of a particle of very small mass in the gravitational field of two heavy bodies (like e.g. the Sun and the Earth). These two *primaries* move in a plane counterclockwise on circles about their common center of mass with the same constant angular velocity. One assumes that the third body moves in the same plane and does not influence the motion of the primaries while it is only influenced by the gravitational forces of the primaries.

In a normalized rotating coordinate system the origin is the center of mass and the two primaries are fixed on the x-axis at $(-\mu, 0)$ and $(1-\mu, 0)$ respectively, where $\mu = m_1/(m_1+m_2)$ and m_1 and m_2 are the masses of the primaries. In this paper we are considering the two systems Sun-Earth-Spacecraft and Sun-Venus-Spacecraft with μ-values of

$$\mu_{SE} = 3.04041307864 \cdot 10^{-6} \text{ and } \mu_{SV} = 2.44770642702 \cdot 10^{-6},$$

respectively.

The equations of motion for the spacecraft with position (x_1, x_2) in rotating coordinates are given by

$$\ddot{x}_1 - 2\dot{x}_2 = \Omega_{x_1}(x_1, x_2), \qquad \ddot{x}_2 + 2\dot{x}_1 = \Omega_{x_2}(x_1, x_2) \tag{1}$$

with

$$\Omega(x_1, x_2) = \frac{x_1^2 + x_2^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2}$$

and

$$r_1 = \sqrt{(x_1+\mu)^2 + x_2^2}, \qquad r_2 = \sqrt{(x_1 - 1 + \mu)^2 + x_2^2}.$$

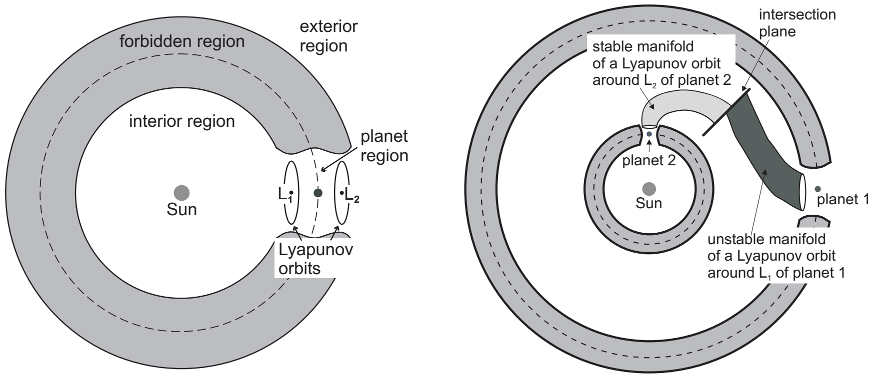
$\Omega_{x_1}, \Omega_{x_2}$ are the partial derivatives of Ω with respect to the variables x_1, x_2.

The equations (1) have a first integral, the *Jacobi integral*, given by

$$C(x_1, x_2, \dot{x}_1, \dot{x}_2) = -(\dot{x}_1^2 + \dot{x}_2^2) + 2\Omega(x_1, x_2). \tag{2}$$

The system posesses five equilibrium points (the *Lagrange points*): the collinear points L_1, L_2 and L_3 on the x-axis and the equilateral points

L_4 and L_5. The three-dimensional manifolds of constant C-values are invariant under the flow of (1), their projection onto position-space, the *Hill's region*, determines the allowed region for the motion of the spacecraft (cf. Figure 1(a)).



(a) (b)

Figure 1. (a) Projection of an energy surface onto position space (schematic) for a value of the Jacobi integral for which the spacecraft is able to transit between the exterior and the interior region. (b) Sketch of the "coupled 3-body approach" (cf. (Koon et al., 2000b, Koon et al., 2002)). The idea is to travel within certain invariant manifold "tubes" possibly including an impulsive maneuver at the intersection plane.

3. Coupling 3-Body Problems

The idea of coupling 3-body problems essentially relies on two key observations:

1. For suitable energy values (i.e. values of the Jacobi integral (2)) there exist periodic solutions, the *Lyapunov orbits* (cf. Figure 1(a)), of (1) in the vicinity of the equilibrium points L_1 and L_2 that are unstable in both time directions. Their unstable resp. stable manifolds W^u resp. W^s are (topologically) cylinders that locally partition the three dimensional energy surface into two sets: (1) transit orbits, that pass between the *interior region* and the *planet region* in the case of an L_1-Lyapunov orbit or between the *exterior region* and the *planet region* in the case of L_2, and (2) non-transit orbits that stay in the exterior or interior region (McGehee, 1969; Koon et al., 2000a).

2. By "embedding" one PCR3BP into a second one, parts of the unstable manifold of a Lyapunov orbit in one system may come close

to the stable manifold of a Lyapunov orbit in the other system (where, for a moment, it may help to imagine that the two systems do not move relative to each other), cf. Figure 1(b). It may thus be possible for a spacecraft to "bridge the gap" between two pieces of trajectories in the vicinity of these manifolds by excerting an impulsive maneuver (Koon et al., 2000b; Koon et al., 2002).

One way to detect a close approach of two such invariant manifolds is to reduce the dimensionality of the problem. One computes the intersection of the two manifolds with a suitable intersection plane (cf. Figure 1(b)) and determines points of close approach in this surface – for example by inspecting projections onto 2D-coordinate planes. This approach has in fact been used for a systematic construction of trajectories that follow prescribed itineraries around and between the Jovian moons (Koon et al., 2002).

4. A Controlled Three Body Problem

In current mission concepts, like for the ESA interplanetary mission *BepiColombo* to Mercury and the current *Smart I* mission, ion propulsion systems are being investigated that continuously exert a small force on the spacecraft ("low-thrust propulsion"). The planar circular restricted three body problem (1) does not model this continuous thrusting capability and the model needs to be enhanced by a suitably defined control term. Here we will restrict our considerations to the special case of a control force whose direction is defined by the spacecraft's velocity, since it is necessary that the acceleration and velocity vectors are parallel for the force to have a maximum impact onto the kinetic energy of the spacecraft. The control term which is to be included into the model is therefore parametrized by a single real value u, determining the magnitude of the control acceleration. We do not take into account that the mass of the spacecraft changes during its flight.

The velocity vector of the spacecraft has to be viewed with respect to the inertial coordinate system and not the rotating one. In view of this, one is lead to the following control system, modeling the motion of the spacecraft under the influence of its low thrust propulsion system in rotating coordinates (cf. Figure 2):

$$\ddot{x} + 2\dot{x}^{\perp} = \nabla\Omega(x) + u\,\frac{\dot{x} + \omega x^{\perp}}{\|\dot{x} + \omega x^{\perp}\|}. \qquad (3)$$

Here, $u = u(t) \in [u_{min}, u_{max}] \subset \mathbb{R}$ denotes the magnitude of the control force, $x = (x_1, x_2)$, $x^{\perp} = (-x_2, x_1)$ and ω is the common angular velocity of the primaries.

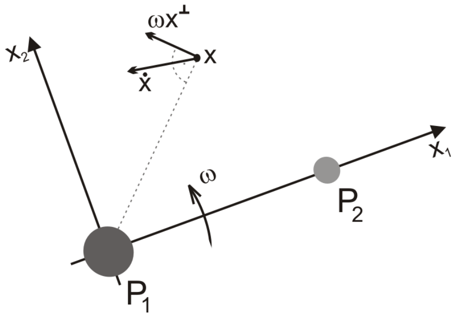


Figure 2. The velocity of the spacecraft with respect to the inertial frame is given by $\dot{x} + \omega x^{\perp}$.

In a mission to Venus the spacecraft will get closer to the Sun, meaning that part of its potential energy with respect to the Sun will be transformed into kinetic energy. As a consequence, the spacecraft's velocity will have to be reduced during its flight such that it matches the one of Venus. Thus, in our concrete application the control values u will actually be negative.

5. Coupling Controlled 3-Body Problems

Obviously, every solution of (1) is also a solution of (3) for the control function $u \equiv 0$. We are going to exploit this fact in order to generalize the coupled 3-body approach as described in Section 3 to the case of controlled 3-body problems. We are still going to use the L_1- and L_2-Lyapunov orbits as "gateways" for the transition between the interior, the planet and the exterior regions. However, instead of computing the relevant invariant manifolds of these periodic orbits, we compute certain *reachable sets* (see e.g. (Colonius and Kliemann, 2000)), i.e. sets in phase space that can be accessed by the spacecraft when employing a certain control function.

Reachable sets. We denote by $\phi(t, x, u)$ the solution of the control system (3) for a given initial point x and a given admissible control function $u \in \mathcal{U} = \{u : \mathbb{R} \to [u_{min}, u_{max}], u \text{ admissible}\}$. Here $u_{min}, u_{max} \in \mathbb{R}$ are predetermined bounds on the magnitude of the control force, and the attribute admissible alludes to the fact that only a certain subset of functions is allowed. Both the bounds and the set of admissible control functions will be determined by the design of the thrusters. For example, the set of admissible control functions could be the set of piecewise constant functions, where the minimal length of an interval on which the function is constant is determined by how fast the magnitude of the accelerating force can be changed within the thrusters.

For a set S in phase space and a given function $\tau : S \times \mathcal{U} \to \mathbb{R}$, we call

$$\mathcal{R}(S,\tau) = \{\phi(\tau(x,u),x,u) \mid u \in \mathcal{U}, x \in S\}$$

the set which is (τ-)*reachable from* S. Later on, we will choose $\tau(x,u)$ in such a way that the reachable sets are contained in the intersection plane.

Coupled controlled 3-body systems. The idea is, roughly speaking, to mimic the coupled 3-body approach while replacing the invariant manifolds of the Lyapunov orbits by certain reachable sets. We exemplarily describe the approach by considering a mission from an outer planet (e.g. Earth) to an inner planet (e.g. Venus).

For two suitable sets $\mathcal{O}_1$ and $\mathcal{O}_2$ (in the vicinity of an L_1–Lyapunov orbit of the Earth and an L_2–Lyapunov orbit of Venus, respectively) one computes associated reachable sets $\mathcal{R}(\mathcal{O}_1,\tau_1) \subset \Sigma_1$ and $\mathcal{R}(\mathcal{O}_2,\tau_2) \subset \Sigma_2$ within suitably chosen intersection planes Σ_1 and Σ_2 in each system, respectively. After a transformation of one of these reachable sets into the other rotating system, the intersection of them is determined. We will describe efficient methods that allow to compute an outer covering of this intersection in Section 6.

Abstractly, the procedure can be summarized as follows:

1. Identify suitable sets $\mathcal{O}_1$ and $\mathcal{O}_2$ in the phase space of the two 3-body problems, respectively. They should be chosen such that all points in $\mathcal{O}_1$ belong to trajectories that transit from the planet region into the interior region of the Earth and those in $\mathcal{O}_2$ transit from the exterior region to the planet region of Venus. Furthermore, in each of the two 3-body problems, choose an intersection plane $\Sigma_i = \{\theta = \theta_i\}$, where $(r,\theta) = (r(x),\theta(x))$ are polar coordinates for the position x of the spacecraft and θ_i is a suitable angle (see also step 3).

2. For points $x_1 \in \mathcal{O}_1$ and $x_2 \in \mathcal{O}_2$ and an admissible control function u, let

$$\begin{aligned} \tau_1(x_1,u) &= \inf\{t > 0 \mid \phi(t,x_1,u) \in \Sigma_1\} \quad \text{and} \\ \tau_2(x_2,u) &= \sup\{t < 0 \mid \phi(t,x_2,u) \in \Sigma_2\}. \end{aligned}$$

   For $i = 1,2$, compute

$$\mathcal{R}(\mathcal{O}_i,\tau_i) = \{\phi(\tau_i(x,u),x,u) \mid x \in \mathcal{O}_i, u \in \mathcal{U}\} \subset \Sigma_i. \qquad (4)$$

3. In order to transform one of the reachable sets $\mathcal{R}(\mathcal{O}_1, \tau_1)$ or $\mathcal{R}(\mathcal{O}_2, \tau_2)$ into the other rotating frame, let $\theta(t)$ be the phase angle between the two planets as seen in the rotating frame of the inner planet. We need to choose a time t_0 such that $\theta(t_0) = \theta_1 - \theta_2$. Particularly, one can consider t_0 to be the time when the spacecraft arrives at the intersection plane. Using t_0 to transform $\mathcal{R}(\mathcal{O}_1, \tau_1)$ into the rotating frame of the inner planet, we obtain the set $\hat{\mathcal{R}}(\mathcal{O}_1, \tau_1) \subset \Sigma_2$. Note that here we exploit the fact that both systems are autonomous.

4. Compute the intersection

$$\hat{\mathcal{R}}(\mathcal{O}_1, \tau_1) \cap \mathcal{R}(\mathcal{O}_2, \tau_2) \subset \Sigma_2 \tag{5}$$

(see Section 6). If this intersection turns out to be empty, typically one needs to increase the range $[u_{min}, u_{max}]$ of the control functions or to choose the section angles θ_1, θ_2 differently. By construction, for each point $x \in \hat{\mathcal{R}}(\mathcal{O}_1, \tau_1) \cap \mathcal{R}(\mathcal{O}_2, \tau_2)$, there exist admissible control functions u_1 and u_2 and times $t_1 = -\tau_1(x, u_1)$, $t_2 = -\tau_2(\tilde{x}, u_2)$, such that $\phi(t_1, \tilde{x}, u_1) \in \mathcal{O}_1$ and $\phi(t_2, x, u_2) \in \mathcal{O}_2$, where $\tilde{x}$ are the coordinates of x with respect to the rotating frame of the outer planet at the phase angle $\theta(t_0)$ between the two planets. Thus, by construction of the sets $\mathcal{O}_1$ and $\mathcal{O}_2$ we have found a controlled trajectory that transits from the outer planet region to the inner planet region.

6. Implementation

Computing the reachable sets. For the purpose of this paper we restrict ourselves to constant control functions and choose a grid of control values in $[u_{min}, u_{max}]$. Although very simple, already this leads to quite satisfactory results. For each of the values on this grid, we numerically integrate the control system (3) by an embedded Runge-Kutta scheme with adaptive stepsize control as implemented in the code DOP853 by Hairer, Nørsett and Wanner, see (Hairer et al., 1993). After each integration step, we check whether the computed trajectory has crossed the intersection plane under consideration and, if this is the case, start Newton's method in order to obtain a point in the section. We store that point, together with the corresponding control value.

Computing the intersection. In step 4 of the algorithm we need to compute the intersection $\hat{\mathcal{R}}(\mathcal{O}_1, \tau_1) \cap \mathcal{R}(\mathcal{O}_2, \tau_2) \subset \Sigma_2$ of the two reachable sets in a common section. We use a *set oriented approach* in order to compute this intersection. Roughly speaking, we construct coverings of $\hat{\mathcal{R}}(\mathcal{O}_1, \tau_1)$ and $\mathcal{R}(\mathcal{O}_2, \tau_2)$ by collections of subsets of Σ_2 and identify those subsets that belong to both coverings. This approach has been used before in the context of the detection of connecting orbits in parameter dependent ordinary differential equations, see (Dellnitz et al., 2001).

More precisely, let $\mathcal{P} = \{P_1, \ldots, P_p\}$ be a finite partition of some relevant bounded part of Σ_2 determined by the region between the two planets under consideration. We compute

$$\begin{aligned} \mathcal{P}_1 &= \{P \in \mathcal{P} \mid \hat{\mathcal{R}}(\mathcal{O}_1, \tau_1) \cap P \neq \emptyset\} \quad \text{and} \\ \mathcal{P}_2 &= \{P \in \mathcal{P} \mid \mathcal{R}(\mathcal{O}_2, \tau_2) \cap P \neq \emptyset\} \end{aligned}$$

and finally

$$\mathcal{T}_{1,2} = \mathcal{P}_1 \cap \mathcal{P}_2.$$

By construction, the set $\bigcup_{P \in \mathcal{T}_{1,2}} P$ contains the intersection $\hat{\mathcal{R}}(\mathcal{O}_1, \tau_1) \cap \mathcal{R}(\mathcal{O}_2, \tau_2)$.

We refer to (Dellnitz and Hohmann, 1997; Dellnitz and Hohmann, 1996; Dellnitz and Junge, 1999) for a detailed description on how to efficiently implement the partitions. In our implementation, when computing $\mathcal{P}_1$ and $\mathcal{P}_2$, in each set $P \in \mathcal{P}_1$ or $P \in \mathcal{P}_2$ we additionally store the minimal ΔV that is necessary to reach this set P from either $\mathcal{O}_1$ or $\mathcal{O}_2$. Whenever the intersection $\mathcal{T}_{1,2}$ consists of more than one partition element, this enables us to choose trajectories with a minimal ΔV (with respect to the chosen control range, intersection planes and gateway sets).

7. On Target for Venus

We apply our method for the construction of continuously controlled interplanetary trajectories to the design of a mission to Venus. In 2005, the European Space Agency will launch *VenusExpress*, a mission to Venus that sends a *MarsExpress*-like spacecraft into an elliptical orbit around Venus via a sequence of impulsive thrust maneuvers. The transfer time from the Earth is around 150 days, while the required ΔV amounts to roughly 1500 m/s ((ESA, 2001; Fabrega et al., 2003)). The interplanetary low-thrust orbit that we are going to construct in this section corresponds to a flight time of roughly 1.4 years, applying a ΔV of approximately 3300 m/s. Since typical low-thrust propulsion systems (as in the ESA mission *Smart I* and the planed cornerstone mission *BepiColombo* for example) have a specific impulse which is approximately one order of magnitude larger than the one of chemical engines, these figures amount to a dramatic decrease in the amount of on-board fuel: at the expense of roughly the 3-fold flight time the weight of the fuel can be reduced to at least 1/3 of what is used for *VenusExpress*.

Computational details. We are now going to comment on the specific details of the computation for the Earth-Venus transfer trajectory, cf. Section 5.

1. For the construction of the 'gateway set' $\mathcal{O}_1$ we consider the L_1-Lyapunov orbit $\mathcal{L}_1$ associated with the value $C_1 = 3.0005$ of the Jacobi integral in the Sun-Earth PCR3BP. This value results from experimenting with several different values and eventually bears further optimization potential. We compute the intersection A_1 of its interior local unstable manifold (i.e. the piece of its local unstable manifold that extends into the interior region) with the section $\Gamma = \{x_1 = 0.98\}$ in the given energy surface $\{C = C_1\}$. Let $\bar{A}_1$ denote the points that are enclosed by the closed curve A_1 in this two-dimensional surface. We set

$$\mathcal{O}_1 = \mathcal{L}_1 \cup (\bar{A}_1 \backslash A_1).$$

   Analogoulsy, we compute $A_2, \bar{A}_2$ and $\mathcal{O}_2$ in the Sun-Venus system, using again a value of $C_2 = 3.0005$ for the Jacobi integral. As intersection planes we choose $\Sigma_1 = \Sigma_2 = \{\theta = \frac{\pi}{4}\}$ – since this turned out to yield the best compromise between transfer time and Δv.

2. We have been using constant control functions only, employing 800 mN as an upper bound for the maximal thrust. This bound is in accordance with the capabilities of the thrusters that are planned to be used in connection with the *BepiColombo* mission. Here we assumed a mass of 4000 kg for the spacecraft.

3./4. Figure 3 shows coverings of the sets $\hat{\mathcal{R}}(\mathcal{O}_1, \tau_1)$ and $\mathcal{R}(\mathcal{O}_2, \tau_2)$, as well as a covering of their intersection $\mathcal{T}_{1,2}$, projected onto the $(x_1, \dot{x}_1)$-plane. The associated trajectory is shown in Figure 4 – in the inertial frame as well as in both rotating frames. It requires a (constant) control force of $u_1 = -651$ mN in the first phase (i.e. while travelling from $\mathcal{O}_1$ to Σ_1) and of $u_2 = -96$ mN in the second phase. The corresponding flight times are $\tau_1 = 0.51$ and $\tau_2 = 0.92$ years, amounting to a total Δv of approximately 3300 m/s. Note that there still exists a discontinuity in the computed trajectory when switching from the first to the second phase. This is due to

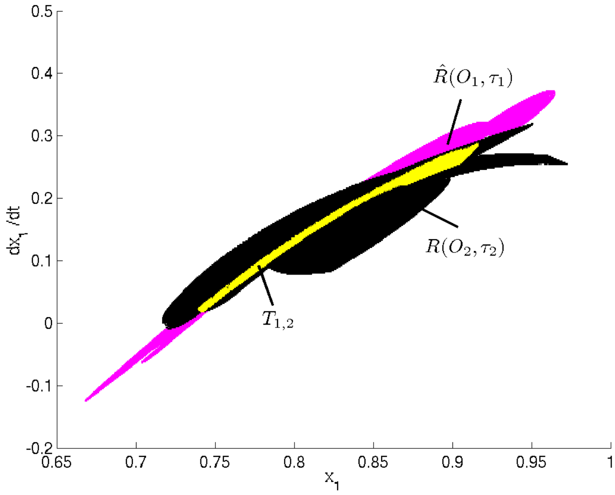
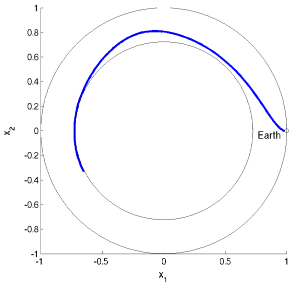


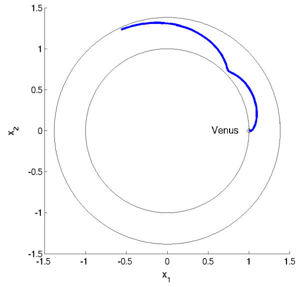
Figure 3. Intersection $\mathcal{T}_{1,2}$ (light grey) of two reachable sets in a common intersection plane. $\hat{\mathcal{R}}(\mathcal{O}_1, \tau_1)$ (dark grey): reachable set of the gateway set of the Earth, $\mathcal{R}(\mathcal{O}_2, \tau_2$ (black): reachable set of the gateway set of Venus. The figure shows a projection of the covering in 3-space onto the $(x_1, \dot{x}_1)$-plane (normalized units).

the fact that the two pieces of the trajectory are only forced to end in the same box in the intersection plane. However, the radii of the boxes are rather small, namely roughly 10 000 km in position space and ≈ 35 m/s in the velocity coordinates. Therefore, we expect the computed trajectory to be a very good initial guess for a standard local solver for a suitably formulated optimal control problem (like, e.g., a collocation or multiple shooting approach, see (Deuflhard et al., 1976; von Stryk, 1993; Stoer and Bulirsch, 2002; Deuflhard and Bornemann, 2002)).

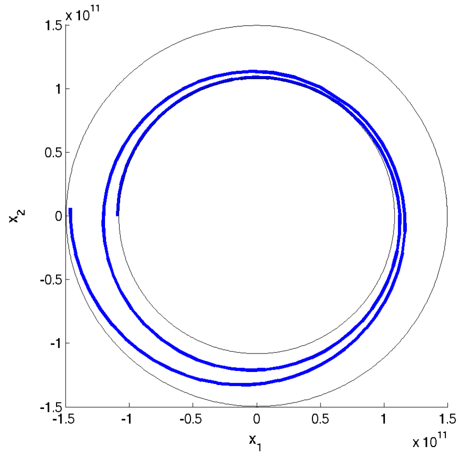
Linking in the Planets. So far, our construction comprised the computation of pieces of controlled trajectories linking the two gateway sets $\mathcal{O}_1$ and $\mathcal{O}_2$ in the neighborhood of the Sun-Earth L_1 and the Sun-Venus L_2 Lagrange points. While missions like the *Genesis discovery mission* (Lo et al., 2001) have shown that one might reach these sets at the expense of very little fuel, it would be interesting to get at least a rough estimate on the flight time and the corresponding Δv for the transfers between the planets and the gateway sets in our case. In particular, it might be worthwhile to find a compromise between flight time and Δv.



(a) Rotating frame of the Earth (normalized units)



(b) Rotating frame of Venus (normalized units)

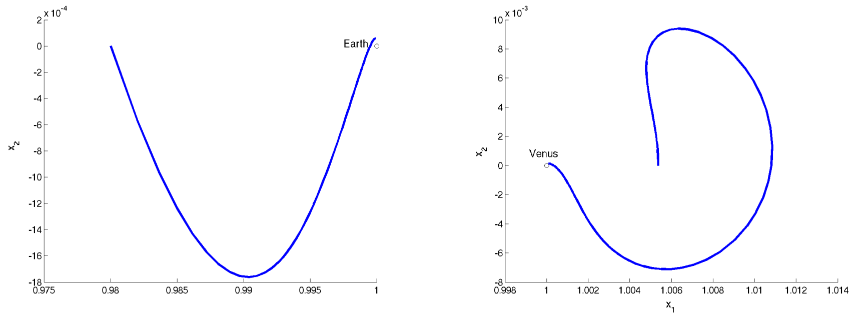


(c) Inertial frame (coordinates in m)

Figure 4. Approximation on an interplanetary trajectory: joining the gateway sets $\mathcal{O}_1$ (near the Sun-Earth L_1) and $\mathcal{O}_2$ (near the Sun-Venus L_2).

To this end we computed extensions of the trajectories between the gateway sets into regions around Earth and Venus, respectively. The main idea of this approach can be found in (Gómez et al., 2001).

From Earth to $\mathcal{O}_1$. The starting point $x_1 \in \mathcal{O}_1$ of the interplanetary patched trajectory computed in the previous section is contained in $\bar{A}_1 \setminus A_1$, i.e. lies "within" the local unstable manifold tube of $\mathcal{L}_1$ which extends into the interior region. Using x_1 as initial value and the prescribed range $[-800, 0]$ mN of control forces, we computed the associated trajectories of the controlled Sun-Earth PCR3BP backward in time until they crossed the section $\Sigma_E = \{x_1 = 1-\mu-10^{-4}\}$. It turns out that for control values between -800 mN and -650 mN these trajectories approach the Earth up to a distance of approximately 15 000 km which we considered close enough for our purposes. Figure 5(a) shows the trajectory for $u_1 = -700\,\mathrm{mN}$, requiring a ΔV of $630\,\mathrm{m/s}$ and a flight time of slightly more than 0.1 years.



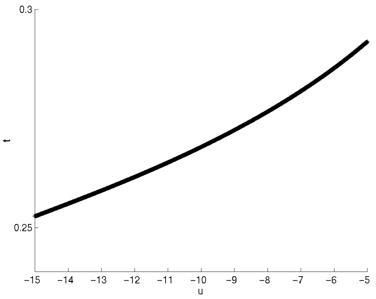
(a) From Earth to the gateway set $\mathcal{O}_1$, control force $-700\,\mathrm{mN}$, flight time 0.1 years, $\Delta V = 630\,\mathrm{m/s}$.

(b) From the gateway set $\mathcal{O}_2$ to Venus, control force $-9\,\mathrm{mN}$, flight time 0.27 years, $\Delta V = 20\,\mathrm{m/s}$.

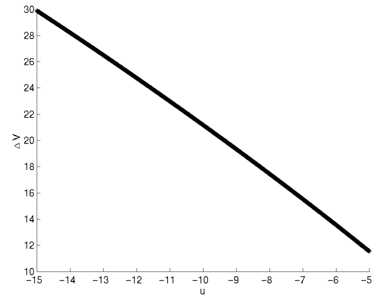
Figure 5. Controlled transit between the gateway sets $\mathcal{O}_1$ and $\mathcal{O}_2$ and the planets (projection onto configuration space, normalized units in the Sun-Earth rotating frame).

From $\mathcal{O}_2$ to Venus. The endpoint $x_2 \in \mathcal{O}_2$ of the interplanetary patched trajectory actually lies on the stable manifold (i.e. on the part that locally extends into the exterior region) of the relevant Sun-Venus L_2-Lyapunov orbit $\mathcal{L}_2$. The transfer from $\mathcal{L}_2$ to Venus is almost free, however, one can slightly decrease the transfer time by employing a small control. Figure 5(b) shows one possible trajectory from $\mathcal{L}_2$ into a $10\,000$ km neighborhood of Venus, employing a control force of $-9\,\mathrm{mN}$, yielding a ΔV of approximately $20\,\mathrm{m/s}$ and a transfer time of 0.27 years. In this case, we used a cross section $\Sigma_V = \{x_1 = 1 - \mu + 10^{-4}\}$. Additionally Figure 6 shows how the transfer time and the required ΔV for this piece of the trajectory depend on the applied control force.

The complete journey. Choosing $u_1 = -700\,\mathrm{mN}$ for the transfer from Earth to the gateway set $\mathcal{O}_1$ and $u_2 = -9\,\mathrm{mN}$ for the trajectory from $\mathcal{O}_2$ to Venus, we finally end up with a flight time of roughly 1.8 years and a corresponding ΔV of slightly less than $4000\,\mathrm{m/s}$ for the complete journey from Earth to Venus. Again, note that these are rough estimates and that the patched trajectory that we constructed should be viewed as an initial guess for a local solver that uses the full model of the solar system.



(a)



(b)

Figure 6. Transfer time t (years) and ΔV (m/s) in dependence of the applied control force u (mN) for the transit from the gateway set $\mathcal{O}_2$ into a neighborhood of Venus.

8. Conclusion

This paper advocates a new approach to the construction of interplanetary low-thrust trajectories. It bases on a recently developed technique for the design of energy efficient trajectories that exploits the structure of the stable and unstable manifolds of certain periodic orbits in the vicinity of the L_1 and L_2 Lagrange points in the circular restricted three body problem. We incorporated a continuously applied, typically small force into the model and showed how one can generalize the concept of invariant manifolds to this context by employing the notion of reachable sets. In combination with set oriented numerical techniques for the efficient computation of the intersection of two suitable reachable sets we constructed an approximate low-thrust trajectory from Earth to Venus that uses a ΔV of approximately 4000 m/s, while requiring a flight time of roughly 1.8 years.

Preliminary computations have shown that a naive extension of our approach to a mission to Mercury does not yield a competitive ΔV in comparision to current mission concepts for the upcoming ESA cornerstone mission *BepiColombo.* Additionally, the model may not be adequate because of the eccentricity and the ecliptic angle of Mercury's orbit.

Our technique has two particular advantages: The first one is that no potentially risky swing-by maneuvers are required and the second one is that the approach inherently provides an increased flexibility with respect to the launch date of the mission. This is due to fact that in principle a spacecraft can stay some time near a periodic orbit such that a required launch date can be met. Furthermore, our approach leads to many trajectories with different flight times.

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References

Abraham, R. and Marsden, J. (1978). *Foundations of Mechanics*. Second Edition, Addison-Wesley.

Colonius, F. and Kliemann, W. (2000). *The dynamics of control*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA. With an appendix by Lars Grüne.

Dellnitz, M. and Hohmann, A. (1996). The computation of unstable manifolds using subdivision and continuation. In Broer, H., van Gils, S., Hoveijn, I., and Takens, F., editors, *Nonlinear Dynamical Systems and Chaos*, pages 449–459. Birkhäuser, *PNLDE* 19.

Dellnitz, M. and Hohmann, A. (1997). A subdivision algorithm for the computation of unstable manifolds and global attractors. *Numerische Mathematik*, 75:293–317.

Dellnitz, M. and Junge, O. (1999). On the approximation of complicated dynamical behavior. *SIAM Journal on Numerical Analysis*, 36(2):491–515.

Dellnitz, M., Junge, O., and Thiere, B. (2001). The numerical detection of connecting orbits. *Discrete Contin. Dyn. Syst. Ser. B*, 1(1):125–135.

Deuflhard, P. and Bornemann, F. (2002). *Scientific computing with ordinary differential equations*, volume 42 of *Texts in Applied Mathematics*. Springer-Verlag, New York.

Deuflhard, P., Pesch, H.-J., and Rentrop, P. (1976). A modified continuation method for the numerical solution of nonlinear two-point boundary value problems by shooting techniques. *Numer. Math.*, 26(3):327–343.

ESA (2001). Venus express mission definition report. *European Space Agency, ESA-SCI*, 6.

Fabrega, J., Schirmann, T., Schmidt, R., and McCoy, D. (2003). Venus express: The first european mission to venus. *International Astronautical Congress*, IAC-03-Q.2.06:1–11.

Gómez, G., Jorba, À., Simó, C., and Masdemont, J. (2001). *Dynamics and mission design near libration points. Vol. III*, volume 4 of *World Scientific Monograph Series in Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ.

Hairer, E., Nørsett, S. P., and Wanner, G. (1993). *Solving ordinary differential equations. I*, volume 8 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, second edition. Nonstiff problems.

Koon, W., Lo, M., Marsden, J., and Ross, S. (2000a). Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics. *Chaos*, 10:427–469.

Koon, W., Lo, M., Marsden, J., and Ross, S. (2000b). Shoot the moon. *AAS/AIAA Astrodynamics Specialist Conference, Florida*, 105:1017–1030.

Koon, W., Lo, M., Marsden, J., and Ross, S. (2002). Constructing a low energy transfer between jovian moons. *Contemporary Mathematics*, 292:129–145.

Lo, M., Williams, B., Bollman, W., Han, D., Hahn, Y., Bell, J., Hirst, E., Corwin, R., Hong, P., Howell, K., Barden, B., and Wilson, R. (2001). Genesis mission design. *Journal of Astronautical Sciences*, 49:169–184.

McGehee, R. (1969). *Some homoclinic orbits for the restricted 3-body problem.* PhD thesis, University of Wisconsin.

Meyer, K. and Hall, R. (1992). *Hamiltonian Mechanics and the n-body Problem.* Springer-Verlag, Applied Mathematical Sciences.

Stoer, J. and Bulirsch, R. (2002). *Introduction to numerical analysis*, volume 12. Springer-Verlag, New York.

Szebehely, V. (1967). *Theory of Orbits – The Restricted Problem of Three Bodies.* Academic Press.

von Stryk, O. (1993). Numerical solution of optimal control problems by direct collocation. In Bulirsch, R., Miele, A., Stoer, J., and Well, K.-H., editors, *Optimal Control - Calculus of Variations, Optimal Control Theory and Numerical Methods*, volume 111 of *Internat. Ser. Numer. Math.*, pages 129–143. Birkhäuser, Basel.