Convergence of some two-level overlapping domain decomposition preconditioners with smoothed aggregation coarse spaces

Caroline Lasser[1] and Andrea Toselli[2]

[1] Center for Mathematical Sciences, TU München, Arcisstr. 21, D-80290 München, Germany. E-mail: classer@mathematik.tu-muenchen.de.
[2] Seminar for Applied Mathematics, ETH Zürich, Rämisstr. 101, CH-8092 Zürich, Switzerland. E-mail: toselli@sam.math.ethz.ch.

Abstract. We study two-level overlapping preconditioners with smoothed aggregation coarse spaces for the solution of sparse linear systems arising from finite element discretizations of second order elliptic problems. Smoothed aggregation coarse spaces do not require a coarse triangulation. After aggregation of the fine mesh nodes, a suitable smoothing operator is applied to obtain a family of overlapping subdomains and a set of coarse basis functions. We consider a set of algebraic assumptions on the smoother, that ensure optimal bounds for the condition number of the resulting preconditioned system. These assumptions only involve geometrical quantities associated to the subdomains, namely the diameter of the subdomains and the overlap. We first prove an upper bound for the condition number, which depends quadratically on the relative overlap. If additional assumptions on the coarse basis functions hold, a linear bound can be found. Finally, the performance of the preconditioners obtained by different smoothing procedures is illustrated by numerical experiments for linear finite elements in two dimensions.

1 Introduction

We consider the scalar Poisson problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega\,, \\ u &= 0, \quad \text{on } \partial\Omega\,, \end{aligned} \tag{1}$$

where Ω is a bounded polyhedral domain in $\mathbb{R}^d$ with $d = 2, 3$.

The discretization of this equation by finite element methods results in a sparse linear system, which is typically too large to be solved directly by Gaussian elimination. Therefore, an iterative solver like the Conjugate Gradient algorithm has to be used. The condition number of the linear system is usually very large and grows quadratically with h^{-1}, where h is the mesh size of the triangulation, thus making convergence very slow. A preconditioner needs to be employed.

Here, we consider a class of two-level overlapping Schwarz preconditioners. These preconditioners consist of two components: the solution of *local* problems associated to an overlapping partition of Ω into subdomains and the solution of a *coarse* problem defined on a low dimensional global space. Local components typically ensure that convergence is independent of h and thus of the size of the original problem (optimality). The coarse level ensures independence of the number of local problems (scalability). Convergence is expected to improve when the relative overlap between the subdomains is increased.

A typical choice for the coarse problem is a finite element approximation on a coarse mesh. For structured meshes, finding a coarse triangulation, such that the fine mesh is a refinement of this coarse one, is relatively easily achieved; see [14] and the references therein. For unstructured meshes, a more general coarse mesh can be employed, but only as long as an interpolation operator from the coarse to the fine finite element space can be found and efficiently implemented; see, e.g., [3]. However, this is not always a trivial task, especially in three dimensions. An alternative approach is realized by smoothed aggregation techniques or partition of unity coarse spaces (see [13]), which provide efficient coarsening procedures without the need of introducing coarse triangulations.

The basic ideas of smoothed aggregation are fairly simple and natural: In a first step, the fine mesh points are aggregated to an initial non-overlapping partition of the domain Ω, and the characteristic functions associated to this non-overlapping partition are considered (aggregation). In a second step, these characteristic functions, whose values typically decrease from one to zero in a layer of width $O(h)$, are smoothed out by the application of a suitable smoothing operator (smoothing). The supports of the smoothed functions define an overlapping partition and corresponding local problems, while their linear span provides a low dimensional coarse problem. This procedure can be applied recursively in order to obtain additional coarse levels for the construction of a multilevel method. The smoother employs the stencil of the finite element matrix and is typically chosen a polynomial of degree $q > 0$ of the original stiffness matrix. The overlap is thus $\delta \sim qh$. The property, that the coarse space represents constant functions, is ensured by exploiting the kernel of the original problem.

An aggregation technique was first introduced in [9] and then quite extensively used for the solution of problems arising in Economics; see [10] and the references therein. Smoothed aggregation techniques have been considered in [18,1] for two-level methods and in [15,17,5,16] for multi-level methods. There, extensive work has been reported on the study of certain smoothers, and practical procedures have been proposed for the initial aggregation, i.e. the initial partition into subdomains. Numerical tests on a large class of scalar and vector problems have been performed. We also mention [8], where smoothed aggregation techniques are applied to discontinuous Galerkin approximations of advection-diffusion problems.

Our assumptions on the smoothers considered here are essentially the same as those already proposed in [18,1]. There, the authors ensure that an optimal preconditioner can be found for the case of generous overlap, i.e., if the overlap δ is comparable to the diameter of the subdomains H. In [7], the case of small overlap is considered, and the condition number of the resulting two-level method is shown to be bounded, if more general assumptions on the overlapping partition and the set of coarse basis functions hold. All these bounds grow quadratically with the inverse of the relative overlap, H/δ. However, a link between the *algebraic* properties of the smoother and optimal bounds for the condition number given in terms of *geometrical* quantities of the overlapping partition is still missing for the case of small overlap. It is the purpose of this paper to bridge this gap. A similar set of assumptions as those given in [18,1] allows us to find the same quadratic bound as in [7]. Using the arguments originally proposed in [13], we also show that if additional assumptions are verified, then a linear bound can be found – as in the case of two-level methods with standard coarse spaces or with partition of unity coarse spaces; see [14, Ch. 5] and [13], respectively. These additional assumptions on the coarse basis functions, however, do not seem to translate into simple algebraic properties on the smoother.

In addition, we enclose some numerical tests in two-dimensions for different choices of the smoother. Although not all of the smoothers tested satisfy the proposed assumptions, our tests do not show any appreciable difference in their numerical performance, i.e., in the number of iterations or the condition number of the preconditioned operator. Our numerical results are consistent with the linear bound on the condition number, even if the additional assumptions on the coarse space required for the proof could not be verified for any of the smoothers considered.

We note that aggregation and smoothed aggregation techniques and partition of unity coarse spaces (see [13]) rely on a similar idea: the coarse basis functions are associated to an overlapping partition into subdomains and no coarse mesh needs to be introduced. However, while in [13] such functions are constructed by assigning explicit nodal values inside the subdomains, in smoothed aggregation techniques they are found by applying a smoother to some initial functions.

The rest of the paper is organized as follows:
In Section 2, we introduce the problem setting and two-level overlapping preconditioners. Section 4 contains the convergence result with quadratic growth in the relative overlap. It is valid, if a suitable set of coarse functions and a proper overlapping partition into subdomains are given. Section 5 deals with their construction by smoothed aggregation techniques. In Section 6, we provide the improved convergence result with linear growth. Section 7 contains the discussion of some smoothing operators, and finally, we present numerical results for a two dimensional problem in Section 8.

2 Problem setting and two-level overlapping preconditioners

We consider the Poisson problem (1). We note, that homogeneous Dirichlet conditions have been chosen just for simplicity, and that more general boundary conditions can be dealt with.

For $u, v \in H^1(\Omega)$, we define the bilinear form

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx.$$

To approximate the solution of (1), we introduce a shape-regular, quasi-uniform triangulation $\mathcal{T}_h$ of Ω, consisting of triangles or tetrahedra. Let h be the maximum of the diameters of its elements. We define

$$V = \{u \in H^1(\Omega) \mid \quad u_{|_\kappa} \in \mathbb{P}_1(\kappa), \ \kappa \in \mathcal{T}_h\},$$

where $\mathbb{P}_1(\kappa)$ is the space of polynomials of maximum degree 1 on κ, and

$$V^0 = V \cap H_0^1(\Omega).$$

Finite element spaces built on quadrilaterals or hexahedra can also be considered, and higher order spaces of piecewise polynomial functions of degree $k > 1$ are possible. The results in this paper remain valid in these cases, with bounds that in general depend on k.

The approximate solution of (1) is then defined as the unique $u \in V^0$ such that

$$A(u,v) = (f,v)_{L^2(\Omega)}, \quad v \in V^0. \tag{2}$$

Problem (2) can be written in matrix form as

$$Au = f. \tag{3}$$

Here, we have used the same notation for a function $u \in V^0$ and the corresponding vector of degrees of freedom, and for a bilinear form $A(\cdot,\cdot)$ and its matrix representation in the space V^0. Similarly, we will use the same notation for functional spaces and the corresponding vector spaces of degrees of freedom.

Next, we introduce a class of two-level overlapping Schwarz preconditioners. Again for simplicity, we only consider additive preconditioners, though multiplicative or hybrid methods can also be devised and analyzed; see [14]. We always assume that we employ exact solvers for the local and coarse problems but approximate solvers could be considered as well. Our theory is easily adjusted to this case; see [14, Ch. 5].

Our preconditioner is uniquely determined by two components:

- an overlapping partition of Ω into subdomains

$$\mathcal{F} = \{\Omega_i' \subset \Omega \mid \quad 1 \leq i \leq N\},$$

which determines the local solvers and ensures optimality;
- a set of coarse basis functions $\{\Phi_i \mid 1 \leq i \leq N\} \subset V^0$, which determines the coarse solver and ensures scalability.

We note that we consider coarse basis functions that are associated to the subdomains. We will make further assumptions in the following.

Given the partition $\mathcal{F}$, the local spaces are defined by

$$V_i = H_0^1(\Omega_i') \cap V^0, \quad 1 \leq i \leq N\,. \tag{4}$$

Let $R_i^T : V_i \to V^0$ be the natural injection operator from the subspace V_i into V^0, which extends a local function by zero to the whole of Ω. We recall, that the restriction operator $R_i : V^0 \to V_i$, defined as the transpose of R_i^T with respect to the Euclidean scalar product, extracts the degrees of freedom inside Ω_i'. The matrix block corresponding to the space V_i is obtained by extracting the degrees of freedom relative to the nodes contained in Ω_i' and is equal to

$$A_i = R_i A R_i^T : V_i \longrightarrow V_i.$$

Our coarse space is defined by

$$V_0 = \text{span}\{\Phi_i,\ 1 \leq i \leq N\}.$$

If $R_0^T : V_0 \to V^0$ is the natural injection operator from the subspace V_0 into V^0, then our coarse solver is

$$A_0 = R_0 A R_0^T.$$

The operators $\{A_i,\ i \geq 0\}$ are symmetric and positive-definite.

The additive Schwarz preconditioner is thus defined as

$$\hat{A}^{-1} = \sum_{i=0}^{N} R_i^T A_i^{-1} R_i,$$

and the corresponding preconditioned operator is

$$P = \hat{A}^{-1} A.$$

3 A convergence result

Optimality and scalability of the Schwarz algorithms introduced in the previous section are not guaranteed for general partitions and coarse spaces without further assumptions. In this section we introduce two sets of sufficient conditions on $\mathcal{F}$ and the coarse functions. They ensure that the resulting additive preconditioner is optimal and scalable and allow to derive quantitative bounds which only involve the relative overlap between the subdomains, as in the case of two-level methods with standard coarse space; see [14, Ch. 5]. We note that most of the content of this section can be found in [1] for the case of generous overlap, and in [7] for the case of small overlap.

Here, we consider functions

$$\{\Phi_i \mid \quad 1 \leq i \leq N\} \subset V^0,$$

such that $\operatorname{supp}\{\Phi_i\} \subset \overline{\Omega_i'}$, $1 \leq i \leq N$. Every function Φ_i vanishes on $\partial\Omega$, and the $\{\Phi_i\}$ are required to form a partition of unity, but only within a proper subset of Ω.

The following two sets of assumptions for the coarse functions $\{\Phi_i\}$ and the partition $\mathcal{F}$ are given in terms of H and δ, $H > \delta > 0$, which reflect the size of the subdomains and the overlap, respectively.

Property 1 (Coarse space I).

1. $|\Phi_i|_1^2 \leq CH^{(d-1)}/\delta$;
2. $\|\Phi_i\|_0^2 \leq CH^d$;
3. There exists $\Omega_{int} \subset \Omega$, such that $\sum_{i=1}^N \Phi_i(x) = 1$ for $x \in \Omega_{int}$, and $\operatorname{dist}(x, \partial\Omega) \leq C\delta$ for $x \in \Omega \setminus \Omega_{int}$;
4. $\operatorname{supp}\{\Phi_i\} \subset \overline{\Omega_i'}$.

We note, that a non-negative function Φ_i, which is constant in the interior of Ω_i' and decreases to zero in a layer of width δ around $\partial\Omega_i'$, satisfies the given bounds for the energy $|\Phi_i|_1^2$ as well as for $\|\Phi_i\|_0^2$. The additional property, that coarse functions must reproduce the constants everywhere except on a layer of width δ around the boundary, will translate into an error estimate for a suitably defined interpolation operator; see Lemma 1.

Property 2 (Partition).

1. $\operatorname{diam}(\Omega_i') \leq CH$;
2. For every $x \in \Omega$, there exists $\Omega_i' \in \mathcal{F}$, such that $x \in \Omega_i'$ and $\operatorname{dist}(x, \partial\Omega_i' \setminus \partial\Omega) \geq c\delta$;
3. There exists C_1 and C_2, such that, for $x \in \Omega$, the ball
$$B(x, rH) = \{y \in \Omega \mid \quad \operatorname{dist}(y, x) \leq rH\}$$
intersects at most $C_1 + C_2 r^d$ subdomains in $\mathcal{F}$;
4. $\operatorname{meas}(\Omega_i) \geq CH^d$.

The first and the last property together ensure that the subdomains have diameter of comparable size H and are shape-regular. According to the second property, δ is a measure of the overlap between the subdomains. The third property is equivalent to the finite covering property, which is standard in overlapping methods; see, e.g., [14, Ch. 5].

The following lemma and its proof can be found in [7, Lem. 2.2].

Lemma 1 (Coarse Interpolant I). *Let Property 1 hold. Then there exists an operator* $Q_0 : H_0^1(\Omega) \to V_0$*, such that*

$$|Q_0 u|_1^2 \leq C \frac{H}{\delta} |u|_1^2,$$
$$\|u - Q_0 u\|_0^2 \leq C H^2 |u|_1^2.$$

To prove a bound for the lowest eigenvalue of the additive operator we need to find a stable decomposition into subspaces, which is given in the following lemma.

Lemma 2. *Let Properties 1 and 2 hold. Then there exists a decomposition* $\{u_i \in V_i,\ 0 \leq i \leq N\}$ *such that*

$$\sum_{i=0}^{N} A(u_i, u_i) \leq C \left(1 + \frac{H}{\delta}\right)^2 A(u,u), \quad u \in V^0.$$

Proof. Given $u \in V^0$, we define

$$u_0 = Q_0 u,$$
$$u_i = I_h(\theta_i(u - u_0)), \quad 1 \leq i \leq N,$$

where I_h is the nodal interpolation operator into the fine mesh, and the family $\{\theta_i\} \subset V$ is a continuous piecewise linear partition of unity relative to $\mathcal{F}$. We recall, that we can find partitions of unity such that

$$\sum \theta_i(x) = 1 \quad x \in \Omega, \qquad \|\theta_i\|_{0,\infty} \leq C, \qquad |\theta_i|_{1,\infty} \leq C/\delta; \tag{5}$$

see, e.g., [14, Pg. 166].

Standard arguments, see [14, Pg. 168], give

$$\sum_{i=1}^{N} |u_i|_{1,\Omega}^2 \leq C \left(\left(1 + \frac{H}{\delta}\right) |u - Q_0 u|_{1,\Omega}^2 + \frac{1}{H\delta} \|u - Q_0 u\|_{0,\Omega}^2 \right).$$

The quadratic bound is then found by applying Lemma 1.

We note, that the previous lemma and its proof have already been given in [7, Lem. 2.3], and that their reformulation here is only meant to motivate the additional assumptions on the coarse basis functions, which will be made in Section 5.

Given Lemma 2 and a coloring argument, we can prove a bound for the condition number of the additive operator; see, e.g., [14, Ch. 5].

Theorem 1. *Let Properties 1 and 2 hold. Then there exist constants* $c_1 > 0$*,* $C_2 > 0$*, such that for all* $u \in V^0$

$$c_1 \left(1 + \frac{H}{\delta}\right)^{-2} A(u,u) \leq A(u, Pu) \leq C_2 A(u,u).$$

Remark 1. We note that the assumption on the triangulation $\mathcal{T}_h$ being quasi-uniform is not employed in the proofs of this section. Indeed, Theorem 1 is a consequence of Properties 1 and 2 for any arbitrary shape-regular mesh $\mathcal{T}_h$.

4 Smoothed aggregation

We now consider the task of finding an overlapping partition and a set of coarse functions that satisfy Properties 1 and 2. We start from an initial partition into non-overlapping subdomains $\mathcal{F}_0 = \{\Omega_i \mid 1 \leq i \leq N\}$. We always assume that these non-overlapping subdomains are shape-regular, and that the diameter of each subdomain is of order H. While algorithms that generate overlapping partitions starting from $\mathcal{F}_0$ can be easily found and implemented, coarse functions that satisfy Property 1 cannot be constructed easily for subdomains of general shape. The method we consider will generate both an overlapping partition and coarse basis functions starting from $\mathcal{F}_0$.

A first choice is to build a coarse space by *aggregation*. We define a set of 'characteristic' functions relative to the initial non-overlapping partition $\mathcal{F}_0$, $\{\Psi_i\} \subset V^0$ and consider the span of these functions. For every node x of $\mathcal{T}_h$ we set

$$\Psi_i(x) = \begin{cases} 0, & x \in \Omega \setminus \Omega_i \text{ or } x \in \partial\Omega, \\ \text{card}(\{j \mid x \in \partial\Omega_j\})^{-1}, & x \in \partial\Omega_i \setminus \partial\Omega, \\ 1, & x \in \Omega_i, \end{cases}$$

where $\text{card}(M)$ denotes the cardinality of a finite set M.

We note, that, if the subdomain boundaries do not contain nodes of the fine mesh $\mathcal{T}_h$, the value of these functions at the nodes is either zero or one, and that they decrease from one to zero in a strip of width h. In the general case they assume values between zero and one, and they decrease from one to zero in a strip of width at most $2h$. Furthermore, the non-vanishing nodal values of Ψ_i cannot be arbitrarily small, since the partition $\mathcal{F}_0$ is shape-regular. These functions form a partition of unity for Ω except in a strip of width $O(h)$ along $\partial\Omega$.

The set $\{\Psi_i\}$ thus satisfies Property 1 with $\delta = h$, and the corresponding coarse space can be analyzed within the framework introduced in the previous section. However, the corresponding additive preconditioner would result in an unsatisfactory bound for the condition number that increases quadratically with H/h. Therefore, the coarse functions $\{\Psi_i\}$ need to be 'smoothed out' to decrease their energy. In order to do so, we apply a suitable operator, called smoother,

$$\Phi_i = S\Psi_i, \quad 1 \leq i \leq N.$$

This smoothing process shall have the effect of increasing the support of the original functions and of creating additional overlap between their supports. We then define the overlapping subdomains by

$$\overline{\Omega_i'} = \text{supp}\{\Phi_i\}, \tag{6}$$

and obtain an overlapping partition $\mathcal{F} = \{\Omega_i' \subset \Omega \mid \ 1 \leq i \leq N\}$.

The smoothing shall also exploit the stencil of the operator A. If $S = p_q(DA)$, where p_q is a polynomial of degree $q \geq 0$ and D a diagonal matrix, then the support of the initial function Ψ_i is increased by q layers of fine elements, which gives an overlap of order $\delta = qh$. In addition, we need to preserve the property that the modified coarse functions $\{\Phi_i\}$ reproduce the constants. This property is guaranteed by the null space of the original differential operator which consists of constant functions. We note that A, the representation of $A(\cdot,\cdot)$ on V^0, is not singular since homogeneous Dirichlet conditions are imposed on $\partial\Omega$, but that when applied to a constant vector, it produces a vector that vanishes everywhere except in a strip around $\partial\Omega$ of width $O(h)$. If $p_q(0) = 1$, then we can write

$$\sum_{i=1}^{N} \Phi_i = S \sum_{i=1}^{N} \Psi_i = S\mathbf{1} = p_q(DA)\,\mathbf{1} = \mathbf{0}' + p_q(0)\mathbf{1} = \mathbf{1}'\,,$$

where $\mathbf{1}$ is the vector of all ones $(1,\ldots,1)^T$, while $\mathbf{0}'$ and $\mathbf{1}'$ are vectors of zeros and ones, respectively, except for entries relative to nodes inside a neighborhood of width $O(qh)$ around $\partial\Omega$. The smoothed coarse functions thus satisfy Property 1.3 with $\delta = qh$.

In view of these remarks, we consider the following assumptions on the initial partition $\mathcal{F}_0$ and the smoother S.

Property 3 (Initial partition and smoother).

1. The initial partition $\mathcal{F}_0$ satisfies
$$cH^d \le \text{meas}(\Omega_i) \le CH^d. \tag{7}$$
2. S is equal to $p_q(DA)$, where p_q is a polynomial of degree q and D a diagonal matrix, such that
   (a) $c\delta \le qh \le C\delta \le C'H$;
   (b) $p_q(0) = 1$;
   (c) $\|S\|_2 \le 1$;
   (d) $\varrho(S^TAS) \le Cq^{-2}\varrho(A)$,
   where $\|\cdot\|_2$ and $\varrho(\cdot)$ denote the spectral norm and the spectral radius of a matrix, respectively.

We note that Properties 3.2.c and 3.2.d have already been considered in [18, Lem. 2.8] and [1, Lem. 4.2] for the case when S is a polynomial in A. A similar property to 3.2.a has been stated in [1, Ass. 4.1], but in terms of the graph corresponding to the initial partition $\mathcal{F}_0$.

Lemma 3. *Let S satisfy Property 3. Then we have for the functions $\Phi_i = S\Psi_i$, $1 \le i \le N$,*
$$\begin{aligned} \|\Phi_i\|_0^2 &\le CH^d, \\ |\Phi_i|_1^2 &\le CH^{d-1}/\delta. \end{aligned} \tag{8}$$

Proof. By construction, the functions Ψ_i and the corresponding column vectors of degrees of freedom, also denoted by Ψ_i, consist of zeros for nodes that belong to elements outside Ω_i. In addition, each non-vanishing entry can be bounded from above and below by a constant. Therefore, $\Psi_i^t \Psi_i$ is bounded from above and below by a constant times the number of nodes inside Ω_i. Since $\mathcal{T}_h$ is shape-regular and quasi-uniform, we have

$$c(\Psi_i^T \Psi_i)\, h^d \leq \text{meas}(\Omega_i) \leq C(\Psi_i^T \Psi_i)\, h^d.$$

If M is the mass matrix, we have

$$\|\Phi_i\|_0^2 = \Psi_i^T S^T M S \Psi_i \leq \varrho(M) \|S\|_2^2 \Psi_i^T \Psi_i \leq C h^d (\Psi_i^T \Psi_i) \leq C\, \text{meas}(\Omega_i),$$

where we have used the property that $\varrho(M)$ is bounded from above by Ch^d; see, e.g., [12, Sect. 6.3.2].

We next consider the second inequality of (8). A trivial unsatisfactory bound can be easily derived, using the fact that $\varrho(A)$ can be bounded from above by Ch^{d-2} (see [12, Sect. 6.3.2]):

$$\|\Phi_i\|_A^2 = \Psi_i^T S^T A S \Psi_i \leq \varrho(S^T A S)(\Psi_i^T \Psi_i) \leq C \frac{h^d\, (\Psi_i^T \Psi_i)}{q^2 h^2} \leq C \frac{H^d}{\delta^2}.$$

To prove a sharper bound, we need to take into account that $AS\Psi_i$ vanishes except in a strip along $\partial\Omega_i'$ of width $O(\delta)$.

For $\kappa \in \mathcal{T}_h$, we denote by R_κ the restriction operator which extracts the degrees of freedom relative to κ and by A_κ the stiffness matrix relative to κ. We note, that if $\overline{\kappa}$ does not intersect $\partial\Omega$, A_κ has a one-dimensional null space consisting of constant functions on κ. We have

$$\|\Phi_i\|_A^2 = \sum_{\kappa \in \mathcal{T}_h} (R_\kappa \Phi_i)^T A_\kappa (R_\kappa \Phi_i).$$

We next define $\Gamma_{i,q}$ as the region of elements, where Φ_i is not constant:

$$\overline{\Gamma_{i,q}} = \bigcup \{\overline{\kappa} \mid \quad \kappa \in \mathcal{T}_h,\ A_\kappa (R_\kappa \Phi_i) \neq 0\}.$$

For $q = 0$, the region $\Gamma_{i,q}$ consists of at most two layers of fine elements, and every application of A adds one additional fine layer in each direction. Therefore, $\Gamma_{i,q}$ is a strip of elements along $\partial\Omega_i'$ of width $O(2qh)$ and thus of measure $O(\delta H^{d-1})$. We clearly have

$$\|\Phi_i\|_A^2 = \sum_{\kappa \subset \Gamma_{i,q}} (R_\kappa \Phi_i)^T A_\kappa (R_\kappa \Phi_i). \tag{9}$$

We next need to relate $R_\kappa \Phi_i = R_\kappa S \Psi_i$ to Ψ_i. We consider an element κ lying in $\Gamma_{i,q}$ and define recursively the regions ω_κ^j, $j \geq 0$. We set $\omega_\kappa^0 = \kappa$ and define ω_κ^j, $j \geq 1$, by taking the union of ω_κ^{j-1} and the neighboring elements that share at least a vertex with ω_κ^{j-1}. Since S is a polynomial of degree q in DA and D is diagonal, the vector $R_\kappa \Phi_i$ is determined only by the values of Ψ_i in ω_κ^q. We set

$$\overline{\Gamma'_{i,q}} = \bigcup_{\kappa \subset \Gamma_{i,q}} \overline{\omega_\kappa^q} \supset \overline{\Gamma_{i,q}},$$

and note that $\Gamma'_{i,q}$ is a layer of elements along $\partial\Omega'_i$ of width $O(4qh)$. Consequently, the expression on the right hand side of (9) is independent of the degrees of freedom of Ψ_i outside the closure of $\Gamma'_{i,q}$. For each node x of $\mathcal{T}_h$, we define

$$\Psi_i^q(x) = \begin{cases} \Psi_i(x), & x \in \overline{\Gamma'_{i,q}}, \\ 0, & \text{otherwise}. \end{cases}$$

Then we can write

$$\|\Phi_i\|_A^2 = (S\Psi_i)^T A (S\Psi_i) = (S\Psi_i^q)^T A (S\Psi_i^q) \leq \varrho(S^T A S)(\Psi_i^q)^T \Psi_i^q.$$

Since $(\Psi_i^q)^T \Psi_i^q$ is bounded from above by the number of nodes contained in the closure of $\Gamma'_{i,q}$, we finally find

$$\|\Phi_i\|_A^2 \leq C \frac{\varrho(A)}{q^2} (\Psi_i^q)^T \Psi_i^q \leq C \frac{h^d \, (\Psi_i^q)^T \Psi_i^q}{q^2 h^2} \leq C \frac{H^{d-1}\delta}{\delta^2} = C \frac{H^{d-1}}{\delta}. \quad (10)$$

Given these bounds on the energy and the norm, respectively, we just need to verify that the functions Φ_i form a partition of unity inside Ω, and that the overlapping partition $\mathcal{F}$ fulfills Property 2. This will be done in the following two lemmas:

Lemma 4. *Let S satisfy Property 3. Then the coarse functions $\Phi_i = S\Psi_i$, $1 \leq i \leq N$, satisfy Property 1.*

Proof. We only need to prove Property 1.3. The function

$$u(x) = \sum_{i=1}^{N} \Psi_i(x)$$

is equal to one at every node x that does not belong to $\partial\Omega$, and consequently at every $x \in \Omega$ outside a strip of width $O(h)$ around $\partial\Omega$. For $k \leq q$, we have

$$((DA)^k u)(x) = 0,$$

except in a strip of width $O(kh)$ around $\partial\Omega$. We thus obtain

$$\sum_{i=1}^{N} \Phi_i(x) = \left(S \sum_{i=1}^{N} \Psi_i \right)(x) = (Su)(x) = 1,$$

at every $x \in \Omega$ outside a strip of width $O(qh) = O(\delta)$ around $\partial\Omega$.

Lemma 5. *Let the initial partition $\mathcal{F}_0$ satisfy Property 3. Then the overlapping partition $\mathcal{F}$ satisfies Property 2.*

Proof. Since the fine mesh $\mathcal{T}_h$ is quasi-uniform, we have

$$\mathrm{diam}(\Omega_i') \leq C(H + qh) \leq CH.$$

Since the original partition is shape-regular, also $\mathcal{F}$ is, and thus we have

$$\mathrm{meas}(\Omega_i') \geq cH^d.$$

We next consider Property 2.2. We first note that for every $x \in \Omega$, there exists i such that $x \in \overline{\Omega_i}$. Since the overlapping subdomains are obtained by adding layers of width $O(qh)$, we have

$$\mathrm{dist}(x, \partial\Omega_i') \geq cqh \geq c\delta.$$

Property 3.1 ensures that, for every i, there is a ball $\mathcal{B}_i \subset \Omega_i$ with $\mathrm{diam}(\mathcal{B}_i) \geq cH$. Since in addition $\mathrm{diam}(\Omega_i') \leq CH$, Property 2.3 holds.

Lemmas 3, 4, and 5, set the stage to apply Theorem 1, and we have:

Theorem 2. *Let Property 3 hold. Then Properties 1 and 2 hold. Therefore, there exist constants c_1, C_2, such that for all $u \in V^0$*

$$c_1 \left(1 + \frac{H}{\delta}\right)^{-2} A(u,u) \leq A(u, Pu) \leq C_2 A(u,u).$$

Remark 2. A closer look at the proofs of this section reveals that the assumption on the global mesh $\mathcal{T}_h$ being quasi-uniform can be relaxed. Indeed it is enough to assume that only the local meshes on the subdomains $\{\Omega_i\}$ are quasi-uniform.

5 An improved convergence result

In this section we find a sharper bound for the condition number of P, which is linear in H/δ. We recall that Schwarz preconditioners with coarse solvers built on coarse triangulations or partition of unity coarse spaces satisfy a linear bound as well. Here, we employ the tools developed in [13]. We recall that there the coarse basis functions are also associated to the subdomains, but are not obtained through smoothed aggregation.

To improve the quadratic bounds of Theorems 1 and 2, we need to impose that our coarse basis functions satisfy additional properties, originally proposed in [13]. In particular they involve bounds on the L^∞-norm of the coarse functions and of their gradients. These additional conditions, however, do not appear to translate into simple conditions on the smoother S, though our numerical results confirm the linear bound, see Section 7.

The algorithm remains the same as before. Only the proof of the decomposition lemma changes, see (5) in Lemma 2 and Lemma 7, and employs the coarse basis functions themselves as the partition of unity $\{\theta_i\}$. Once additional assumptions are satisfied on the coarse basis functions, the proof of the decomposition lemma can be carried out exactly as in [13] and is not presented here.

We consider the same set of coarse functions as before

$$\{\Phi_i \mid \quad 1 \le i \le N\} \subset V^0,$$

and the same coarse space

$$V_0 = \operatorname{span}\{\Phi_i,\ 1 \le i \le N\}.$$

However, we also need an additional function $\Phi_B \in V$ associated to the boundary $\partial\Omega$, so that the augmented set of functions forms a partition of unity on the entire $\overline{\Omega}$. This additional function is only needed for the proof and need not be implemented in practice. Given a partition $\mathcal{F}$, we consider the following assumptions; see [13]:

Property 4 (Coarse space II).

1. $\|\Phi_i\|_{0,\infty} \le C$ for $1 \le i \le N$ and $i = B$;
2. $|\Phi_i|_{1,\infty} \le C/\delta$ for $1 \le i \le N$ and $i = B$;
3. $\sum_{i=B,1}^{N} \Phi_i(x) = 1$ for $x \in \overline{\Omega}$;
4. $\operatorname{supp}\{\Phi_i\} \subset \overline{\Omega_i'}$, $1 \le i \le N$.

The same interpolation operator defined in Lemma 1 can be considered here and the same bounds in Lemma 1 can be proven by noting that Properties 4.1 and 4.2 imply Properties 2.1 and 2.2, respectively.

Lemma 6 (Coarse Interpolant II). *Let Property 4 hold. Then there exists an operator $Q_0 : H_0^1(\Omega) \to V_0$, such that*

$$|Q_0 u|_1^2 \le C \frac{H}{\delta} |u|_1^2,$$
$$\|u - Q_0 u\|_0^2 \le C H^2 |u|_1^2.$$

A stable decomposition can then be found by using the same proof as in [13, Th. 1], since our coarse functions Φ_B and Φ_i satisfy the same properties as θ_B^δ and θ_i^δ in [13]. We thus have

Lemma 7. *Let Properties 4 and 2 hold. Then there exists a decomposition $\{u_i \in V_i,\ 0 \le i \le N\}$ such that*

$$\sum_{i=0}^{N} A(u_i, u_i) \le C \left(1 + \frac{H}{\delta}\right) A(u, u), \quad u \in V^0.$$

Our final result is the following theorem:

Theorem 3. *Let Properties 4 and 2 hold. Then there exist constants* $c_1, C_2 > 0$*, such that, for all* $u \in V^0$*,*

$$c_1 \left(1 + \frac{H}{\delta}\right)^{-1} A(u,u) \leq A(u, Pu) \leq C_2 A(u,u).$$

Remark 3.

1. If $\|S\|_\infty < C$, then Property 4.1 holds. This is easily seen using an analogous argument as in the proof for the corresponding L^2-bound of Lemma 3. However, we were not able to find a suitable condition on S that ensures Property 4.2.
2. The error bound in Lemma 6 is not needed for the proof of Lemma 7.

6 Some choices for the smoother

Most of the choices for the smoother S presented in this section have already been proposed in the literature; see [1], [18], [15], and [2]. Not all the Properties 3.2 can be proved for them, except for the recursive Richardson smoother. However, all of them show comparable iteration counts and condition numbers in our numerical experiments; see Section 7. This is due to the fact that the inequalities in Properties 3.2.c and 3.2.d are asymptotic bounds for large q, which is roughly the number of fine mesh layers of overlap between the subdomains, while in practice q is usually not large at all.

Both, the simple and the recursive Richardson smoother, rely on a known upper bound $\hat{\varrho}$ for the spectral radius $\varrho(A)$, that satisfies

$$\varrho(A) \leq \hat{\varrho} \leq C_\varrho\, \varrho(A). \tag{11}$$

6.1 A simple Richardson smoother

We define

$$S = S_q = (Id - \omega\hat{\varrho}^{-1}A)^q\,, \quad \omega \in (0,1]\,, \quad q \geq 0\,.$$

S is the smoother for Richardson's method with relaxation parameter $\omega\hat{\varrho}^{-1}$, similar to a smoother proposed in [15]. It involves q applications of A and it can be defined recursively by the relation

$$S_q = (Id - \omega\hat{\varrho}^{-1}A)S_{q-1}.$$

Consequently, functions and overlapping subdomains with larger overlap can be calculated from the previous ones. By construction, S is a polynomial p_q of degree q in A with $p_q(0) = 1$. It satisfies Property 3 except 3.2.d. We can only prove a somewhat weaker bound for $\varrho(S^T AS)$ that is of order q^{-1}:

Lemma 8. *Let* $S = (Id - \omega\hat{\varrho}^{-1}A)^q$*,* $\omega \in \mathbb{R}$*, and* $q \geq 0$*. Then we have*

$$\|S\|_2 \leq (\max\{|1-\omega|, 1\})^q \,.$$

If in addition $\omega \in (0,1]$ *and* $q > 0$*, then*

$$\varrho(S^T A S) \leq C_\varrho \frac{\varrho(A)}{2q\omega} \,.$$

Proof. For the first bound it is enough to show that

$$\|S_1\|_2 \leq \max\{|1-\omega|, 1\}.$$

Since S_1 is symmetric, we have to examine its eigenvalues. They satisfy $\lambda(S_1) = 1 - \omega\hat{\varrho}^{-1}\lambda(A)$. Since A is positive definite, we have $\lambda(S_1) \leq 1$. Furthermore,

$$\omega\hat{\varrho}^{-1}\lambda(A) \leq \omega\hat{\varrho}^{-1}\varrho(A) \leq \omega \,,$$

and therefore $\lambda(S_1) \geq 1 - \omega$.

We now consider the second bound. We have $S^T A S = S^2 A$, and obtain for $\omega \in (0,1]$

$$\varrho(S^T A S) \leq \max_{t\in[0,\varrho(A)]} \left|(1-\omega\hat{\varrho}^{-1}t)^{2q}\, t\right| \leq \max_{t\in[0,\hat{\varrho}\omega^{-1}]} \left|(1-\omega\hat{\varrho}^{-1}t)^{2q}\, t\right| \,.$$

The function $f(t) := (1-\omega\hat{\varrho}^{-1}t)^{2q}\, t$ is non-negative in $[0, \hat{\varrho}\omega^{-1}]$ and satisfies $f(0) = f(\hat{\varrho}\omega^{-1}) = 0$. Its maximum is attained for

$$\bar{t} = \frac{\hat{\varrho}}{(2q+1)\omega} \,.$$

Evaluating $f(\bar{t})$ and using $\hat{\varrho} \leq C_\varrho \varrho(A)$ yields the upper bound.

With the same arguments as in the proof of Lemma 3, we can prove, that for a fixed ω the coarse basis functions Φ_i only satisfy

$$|\Phi_i|_1^2 \leq C\frac{H^{d-1}}{h}.$$

As already discussed in Section 4, one would expect that the condition number of the corresponding preconditioner increases quadratically with H/h. However, our numerical experiments show comparable behavior for this simple smoother and the recursive one introduced later, for which the desired sharper bound holds; see Sections 6.2 and 7. This is due to the fact, that in practice q is not large, and thus the following remark applies.

Remark 4. For every $q_0 \geq 0$ there exists a constant $C_0 = C(q_0) > 0$, such that for $0 \leq q \leq q_0$ and $\omega \in (0,1]$

$$\varrho(S^T A S) \leq C_\varrho\, C_0\, \frac{\varrho(A)}{q^2\omega}.$$

This can be easily seen from the proof of the previous lemma. Indeed we have

$$C_0 = \max_{0 \le q \le q_0} \left\{ \left(\frac{2q}{2q+1} \right)^{2q} \frac{q^2}{(2q+1)} \right\}. \tag{12}$$

The key point here is that C_0 remains small for high values of q. We have for instance $C_0 \le 19$ for $q_0 = 100$, which is a value for q that is far larger than those employed in practice.

6.2 A recursive Richardson smoother

We now recall the smoother that was introduced and studied in [1,18]. We define

$$S = S_k = \prod_{i=0}^{k} \left(Id - \frac{4}{3} \varrho_i^{-1} A_i \right), \quad k \ge 0 \tag{13}$$

where $\varrho_i = 9^{-i} \hat{\varrho}$, $i \ge 0$, and

$$A_0 = A\,, \quad A_i = \left(Id - \frac{4}{3} \varrho_{i-1}^{-1} A_{i-1} \right)^2 A_{i-1}\,, \quad i \ge 1.$$

S is a recursive Richardson smoother with prefixed relaxation parameters $4/3\varrho_i^{-1}$. It is a polynomial p_q in A with $p_q(0) = 1$. For the polynomial degree q only certain values are possible. Since $\deg(A_0) = 1$ and $\deg(A_i) = 3\deg(A_{i-1})$, $i \ge 1$, we have $\deg(A_i) = 3^i$, $i \ge 0$. Therefore, $\deg(S_0) = 1$ and

$$\deg(S_k) = \deg(A_k) + \deg(S_{k-1}) = 3^k + \deg(S_{k-1})\,, \quad k \ge 1.$$

Consequently,

$$q = \deg(S_k) = \sum_{i=0}^{k} 3^i = \frac{3^{k+1} - 1}{2}, \quad k \ge 0.$$

The following lemma and its proof can be found in [18, Lem. 2.8]. It shows that S satisfies Property 3.

Lemma 9. *Let S be the smoother defined in (13), and $q = \deg(S)$. Then we have*

$$\|S\|_2 \le 1\,, \quad \varrho(S^T A S) \le \frac{C_\varrho}{4} q^{-2} \varrho(A)\,.$$

6.3 A SPAI smoother

The Richardson smoothers previously introduced depend on a relaxation parameter, which is prefixed in the case of the recursive smoother. However, a preconditioner that does not involve parameters that need to be tuned to the particular type of problem and its coefficients, is preferable; see [8] for the application of smoothed aggregation techniques to an advection-diffusion problem. A parameter-free smoothing operator built with a so-called sparse approximate inverse (SPAI) of the stiffness matrix A is given by

$$S = S_q = (Id - DA)^q, \quad q \geq 0, \tag{14}$$

where the SPAI-matrix D minimizes the Frobenius norm $\|Id - DA\|_F$ over the set of diagonal matrices.

Let n be the size of A. If the column vectors of a matrix M are denoted by m_k, and if e_k is the k-th column of the identity matrix, we write

$$\|Id - DA\|_F^2 = \sum_{k=1}^{n} \|Ad_k - e_k\|_2^2 = \sum_{k=1}^{n} \|d_{kk} a_k - e_k\|_2^2,$$

which is minimized by $D = \operatorname{diag}\{d_{kk}\}$ with

$$d_{kk} = \frac{a_{kk}}{\|a_k\|_2^2}, \quad 1 \leq k \leq n\,;$$

see [2]. If A is the discrete Laplacian with periodic boundary conditions, which results from a standard second-order finite difference discretization, the SPAI smoother of (14) is the standard damped Jacobi smoother with optimal damping parameter ω; see [2, Proposition 1].

The norm of S satisfies a weaker bound than that stated in Property 3.2.c.

Lemma 10. *Let S be the SPAI smoother defined in (14). Let n denote the size of A, and p_i, $1 \leq i \leq n$, the number of nonzero off-diagonal entries in the i-th row of A. Then we have*

$$\|S\|_2 \leq C(p)^q,$$

where $p = \max_{1 \leq i \leq n} p_i$, and $C(p) = \sqrt{(1+p)(1+\sqrt{p})}$.

Proof. It is enough to show $\|S_1\|_2 \leq C(p)$. We will establish bounds for the row-sum-norm of S and S^T, since

$$\|S\|_2 \leq \|S\|_\infty^{\frac{1}{2}} \|S^T\|_\infty^{\frac{1}{2}};$$

see [6, Exercise 2.9.6].

$$\|S\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |s_{ij}| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} \left| \delta_{ij} - \frac{a_{ii} a_{ij}}{\|a_i\|_2^2} \right|$$

$$
\begin{aligned}
&= \max_{1\le i\le n}\left(1-\frac{a_{ii}^2}{\|a_i\|_2^2}+\sum_{j\ne i}\frac{a_{ii}|a_{ij}|}{\|a_i\|_2^2}\right)\\
&\le \max_{1\le i\le n}\left(1-\frac{a_{ii}^2}{\|a_i\|_2^2}+\frac{a_{ii}}{\|a_i\|_2^2}\cdot\|a_i\|_2\cdot\sqrt{p_i}\right)\\
&\le 1+\sqrt{p},
\end{aligned}
$$

where we have employed Cauchy-Schwarz for the last but one inequality, and used the symmetry of A.

Furthermore,

$$
\begin{aligned}
\|S^T\|_\infty &= \max_{1\le i\le n}\sum_{j=1}^{n}|s_{ji}| = \max_{1\le i\le n}\sum_{j=1}^{n}\left|\delta_{ji}-\frac{a_{jj}a_{ji}}{\|a_j\|_2^2}\right|\\
&= \max_{1\le i\le n}\left(1-\frac{a_{ii}^2}{\|a_i\|_2^2}+\sum_{j\ne i}\frac{a_{jj}|a_{ij}|}{\|a_j\|_2^2}\right)\\
&\le \max_{1\le i\le n}\left(1-\frac{a_{ii}^2}{\|a_i\|_2^2}+\sum_{j\ne i}\frac{|a_{ij}|}{\|a_j\|_2}\right) \le \max_{1\le i\le n}\left(1-\frac{a_{ii}^2}{\|a_i\|_2^2}+1\cdot p_i\right)\\
&\le 1+p.
\end{aligned}
$$

Here p is equal to the maximal number of neighbors that a mesh point of $\mathcal{T}_h$ has, and for, e.g., $p=7$, we have $C(p)\sim 5.4$. We note that $C(p)>1$ and that consequently Property 3.2.c cannot be derived from Lemma 10.

If one does not assume A to be weakly diagonal dominant, only triangulations with $p\le 7$ give an upper bound for $\varrho(S^TAS)$ which asymptotically decays to zero for $q\to\infty$; see [2, Theorem 1]) for the proof of the following lemma.

Lemma 11. *Assume $p\le 7$, and denote with*

$$
\hat{C}=\max_{1\le i\le n}\frac{\|a_i\|_2^2}{a_{ii}^2},\qquad \Gamma=\frac{1}{2}(1+\sqrt{1+p})<2.
$$

Then we have

$$
\varrho(S^TAS)\le \hat{C}\,\eta(2q)\,\varrho(A), \tag{15}
$$

where

$$
\eta(2q)=\max\left\{\left(\frac{2q}{2q+1}\right)^{2q}\frac{1}{2q+1},\,\Gamma(\Gamma-1)^{2q}\right\}.
$$

Though the bound in the previous lemma is even weaker than the one obtained for the simple Richardson smoother and, of course, fails to satisfy Property 3.2.d, the numerical performance of the SPAI smoother is comparable to the results obtained by the other choices for S, see Section 7.

7 Numerical results

We have tested the performance of our Schwarz preconditioner for the Poisson problem

$$\begin{aligned} -\Delta u &= xe^y, \quad \text{in } \Omega = (0,1)^2\,, \\ u &= -xe^y, \quad \text{on } \partial\Omega\,. \end{aligned} \tag{16}$$

This choice of Dirichlet conditions also numerically confirms our claim that inhomogeneous boundary conditions can be dealt with.

The mesh is built by dividing Ω into n^2 equal fine squares and cutting them into halves. Thus, we obtain a triangulation $\mathcal{T}_h$ with $h = \frac{1}{n}$, $h \in \left\{\frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}\right\}$. The maximum number of neighbors, which a mesh point in $\mathcal{T}_h$ has, is $p = 6$. The aggregation routine partitions Ω into non-overlapping subsquares Ω_i of area H^2 with $H \in \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}\right\}$.

Depending on the polynomial degree q of the smoother, which varies between 0 and 14 in our experiments, we obtain an overlapping partition $\mathcal{F}$ with overlap $\delta = (q+2)h$. The sum $q+2$ results from the fact, that the boundaries $\partial\Omega_i$ of the non-overlapping subsquares contain fine mesh points, which causes the support of the initial, unsmoothed coarse basis functions Ψ_i overlap in a strip of width $2h$.

For the simple and the recursive Richardson smoother, we replace the upper bound for the spectral radius of the stiffness matrix by an estimate for the spectral radius, which is given by the *Matlab* built-in function *normest.* We use linear finite elements, and solve the resulting linear system by Conjugate Gradient. The Conjugate Gradient routine employed belongs to the NetLib-linalg package, see [11]. It provides an estimate for the condition number of the preconditioned operator by dividing the maximal and the minimal eigenvalue of a suitable tridiagonal matrix, which approximates the preconditioned operator. Our stopping criterion is the reduction of the residual norm by a factor of 10^{-6} or the exceeding of 100 iterations.

Effectiveness of the coarse solve: Tables 1 and 2 show iteration counts and estimated condition numbers for the one- and two-level preconditioners built with the SPAI smoother. As expected, iteration counts and condition numbers for the one-level algorithm increase rapidly with the number of subdomains, while the coarse solve in the corresponding two-level algorithm, see Table 2, only allows a moderate increase or keeps the iterations counts bounded for the case of generous overlap.

Comparison of some smoothers: Tables 2, 3 and 4 show iteration counts and estimated condition numbers of the two-level preconditioned operators which were obtained with the SPAI, the simple Richardson, and the recursive Richardson smoother, respectively. The relaxation parameter for the simple Richardson smoother was chosen as $\omega = 2/3$. Though only the recursive Richardson smoother meets the required theoretical bounds, there is just a slight difference in the numerical performance of the three smoothers. As previously remarked, this can be explained by the fact that only small values of the smoother polynomial degree q are employed in practice, while our theory requires bounds which hold for large q as well.

Linear dependence on the relative overlap: Figure 1 shows a linear and quadratic least-squares fit for the estimated condition number of the preconditioner built with the recursive Richardson smoother versus the relative overlap H/δ. Since the coefficient for the quadratic term is small and the least-square relative error is the same for the linear and the quadratic fit, our experiments confirm a linear dependence.

Table 1. Iteration counts and estimated condition numbers (in parenthesis) for Conjugate Gradient with one-level preconditioner versus h and the relative overlap.

		H/δ			
h^{-1}	H^{-1}	16	8	4	2
16	2	-	-	13 (16.7)	12 (6.0)
16	4	-	-	-	17 (25.4)
32	2	-	17 (32.6)	14 (11.1)	11 (5.3)
32	4	-	-	24 (52.8)	16 (15.3)
32	8	-	-	-	31 (93.2)
64	2	22 (64.6)	17 (21.7)	13 (9.6)	11 (5.0)
64	4	-	34 (107.4)	22 (33.6)	15 (12.5)
64	8	-	-	42 (198.5)	27 (54.9)
64	16	-	-	-	57 (365.5)
128	4	46 (216.7)	30 (70.2)	21 (28.3)	14 (11.4)
128	8	-	59 (408.8)	40 (125.2)	25 (44.2)
128	16	-	-	80 (782.1)	52 (214.1)
128	32	-	-	-	100 (1454.8)

Fig. 1. Linear and quadratic least-squares fit for the estimated condition number of the two-level preconditioned operator versus the relative overlap H/δ. The preconditioner is built using the recursive Richardson smoother.

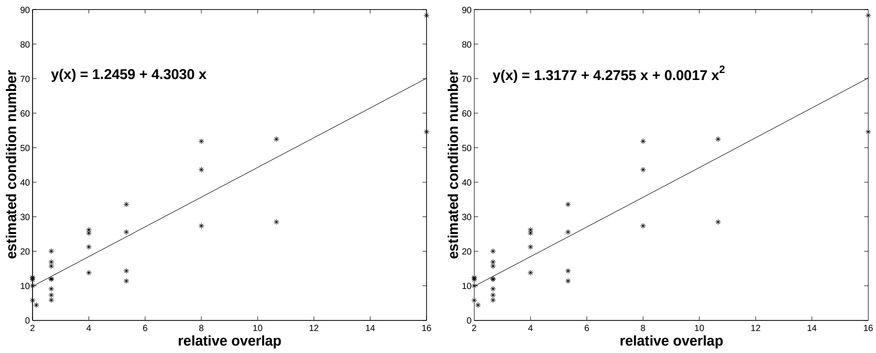

Table 2. Iteration counts and estimated condition numbers (in parenthesis) for Conjugate Gradient with the two–level preconditioner, using the SPAI smoother.

		H/δ			
h^{-1}	H^{-1}	16	8	4	2
16	2	-	-	15 (13.8)	13 (5.1)
16	4	-	-	-	16 (10.0)
32	2	-	18 (27.4)	15 (9.7)	13 (4.8)
32	4	-	-	22 (21.3)	16 (7.8)
32	8	-	-	-	19 (11.8)
64	2	24 (54.6)	19 (19.2)	14 (8.8)	12 (4.8)
64	4	-	30 (43.6)	21 (17.5)	15 (8.0)
64	8	-	-	26 (25.3)	19 (10.5)
64	16	-	-	-	19 (12.3)
128	4	41 (88.3)	28 (36.4)	19 (17.8)	15 (8.7)
128	8	-	38 (51.9)	27 (23.8)	19 (12.3)
128	16	-	-	27 (26.2)	21 (11.6)
128	32	-	-	-	19 (12.4)

Table 3. Iteration counts and estimated condition numbers (in parenthesis) for Conjugate Gradient with the two-level preconditioner, using the simple Richardson smoother.

		H/δ			
h^{-1}	H^{-1}	16	8	4	2
16	2	-	-	15 (13.8)	13 (5.4)
16	4	-	-	-	16 (10.0)
32	2	-	18 (27.4)	15 (10.1)	13 (5.0)
32	4	-	-	22 (21.3)	16 (9.4)
32	8	-	-	-	19 (11.8)
64	2	24 (54.6)	19 (19.9)	14 (9.1)	13 (5.0)
64	4	-	30 (43.6)	22 (20.4)	16 (9.6)
64	8	-	-	26 (25.3)	22 (13.9)
64	16	-	-	-	19 (12.3)
128	4	41 (88.3)	30 (42.0)	20 (20.9)	15 (10.0)
128	8	-	38 (51.9)	30 (30.6)	21 (16.9)
128	16	-	-	27 (26.2)	25 (16.1)
128	32	-	-	-	19 (12.4)

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Table 4. Iteration counts (first rows) and estimated condition numbers (second rows, in parenthesis) of the two-level preconditioner with recursive Richardson smoothing.

		H/δ							
h^{-1}	H^{-1}	16	$10\frac{2}{3}$	8	$5\frac{1}{3}$	4	$2\frac{2}{3}$	$2\frac{2}{15}$	2
16	2	-	-	-	-	15	14	-	-
		-	-	-	-	(13.8)	(7.3)	-	-
16	4	-	-	-	-	-	-	-	16
		-	-	-	-	-	-	-	(10.0)
32	2	-	-	18	17	-	13	-	-
		-	-	(27.4)	(14.3)	-	(5.9)	-	-
32	4	-	-	-	-	22	18	-	-
		-	-	-	-	(21.3)	(12.0)	-	-
32	8	-	-	-	-	-	-	-	19
		-	-	-	-	-	-	-	(11.8)
64	2	24	20	-	15	-	-	11	-
		(54.6)	(28.5)	-	(11.4)	-	-	(4.4)	-
64	4	-	-	30	25	-	16	-	-
		-	-	(43.6)	(25.6)	-	(9.1)	-	-
64	8	-	-	-	-	26	23	-	-
		-	-	-	-	(25.3)	(15.7)	-	-
64	16	-	-	-	-	-	-	-	19
		-	-	-	-	-	-	-	(12.3)
128	4	41	33	-	-	-	22	-	13
		(88.3)	(52.5)	-	-	-	(20.0)	-	(5.8)
128	8	-	-	38	31	-	20	-	-
		-	-	(51.9)	(33.6)	-	(11.9)	-	-
128	16	-	-	-	-	27	24	-	-
		-	-	-	-	(26.2)	(16.9)	-	-
128	32	-	-	-	-	-	-	-	19
		-	-	-	-	-	-	-	(12.4)

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