Overlapping preconditioners for discontinuous Galerkin approximations of second order problems

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1 Introduction

The purpose of this paper is to present a two–level overlapping preconditioner for discontinuous Galerkin finite element discretization of advection-diffusion problems in two or three dimensions. Our problem is discretized using a discontinuous Galerkin (DG) finite element method. The original domain is then subdivided into overlapping subdomains in order to introduce a number of local problems. We propose two different coarse problems. The first one is an advection–diffusion problem discretized using a continuous finite element space on a coarse triangulation. The second employs a smoothed aggregation technique and does not require the introduction of a coarse mesh. The performance of the corresponding two methods is illustrated for two test problems in two dimensions discretized with linear finite elements.

Discontinuous Galerkin approximations have been used since the early 1970s and are recently becoming more and more popular for the approximation of a large class of problems; we refer to [2] for a comprehensive review of these methods. Here, we consider a discontinuous *hp*-finite element method proposed in [4]. As for many DG methods, the approximate solution belongs to a space of discontinuous finite element functions, i.e., it is piecewise polynomial of a certain degree on a given triangulation, being in general discontinuous across the elements. Suitable bilinear forms, which also contain interface contributions, are then employed, in order to ensure consistency.

We only know of two previous works on DD preconditioners for DG approximations; see [3, 6].

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2 Continuous and discrete problems

We consider the following scalar advection-diffusion problem with Dirichlet conditions

$$\begin{aligned} \mathcal{L}u \;=\; -\nabla\cdot(a\nabla u) + b\cdot\nabla u + cu \;&=\; f, \quad \text{in } \Omega\,, \\ u \;&=\; 0, \quad \text{on } \Gamma\,, \end{aligned} \tag{1}$$

where Ω is a bounded open polyhedral domain in R^d, $d = 2, 3$, and Γ its boundary. Problem (1) describes a large class of diffusion-transport-reaction processes.

We consider problem (1) and make some further hypotheses. We assume that $a = \{a_{i,j}\}_{i,j=1}^d$ is a symmetric positive–semidefinite matrix,

$$\xi^T a(x)\xi \geq 0, \quad \xi \in \mathrm{R}^d, \quad x \in \Omega,$$

b and c are a vector field in $W^{1,\infty}(\Omega)$ and a function in $L^\infty(\Omega)$, respectively, such that

$$(c - \frac{1}{2}\nabla\cdot b)(x) \;\geq\; 0\,, \quad x \in \Omega\,, \tag{2}$$

and the right-hand side f is a function in $L^2(\Omega)$. The existence of a unique solution of (1) is shown in [4]. We note that we have considered only the case of strongly–imposed homogeneous Dirichlet boundary conditions for simplicity, but that more general ones can be employed, such as Neumann, Robin, or weakly–imposed Dirichlet conditions. Our methods can be extended to these cases. We also recall that in case a does not have full rank, Dirichlet conditions can only be imposed on a part of the boundary; see [4].

We next introduce $\mathcal{T}_h$, a conforming, shape–regular triangulation of Ω consisting of open simplices κ with diameter $O(h)$. We denote by $\mathcal{P}_k(\kappa)$ the space of polynomials on κ of total degree $k \in \mathrm{N}_0$ and define the vector of local polynomial degrees $\mathbf{p} = (p_\kappa : \kappa \in \mathcal{T}_h)$. We consider the finite element space

$$S^{\mathbf{p}}(\Omega, \mathcal{T}_h) = \{u \in L^2(\Omega) : u|_\kappa \in \mathcal{P}_{p_\kappa}(\kappa)\}\,.$$

and define $S_0^{\mathbf{p}}(\Omega, \mathcal{T}_h)$ as the subspace of functions in $S^{\mathbf{p}}(\Omega, \mathcal{T}_h)$ vanishing on Γ. Our FE approximation space is chosen as $V^h = S_0^{\mathbf{p}}(\Omega, \mathcal{T}_h)$.

We define $\mathcal{E}_{int}$ as the set of edges that are intersections of the element boundaries and Γ_{int} as the union of the edges in $\mathcal{E}_{int}$. For $\kappa \in \mathcal{T}_h$, we then denote the unit outward normal to $\partial\kappa$ at $x \in \partial\kappa$ by $\mu_\kappa(x)$ and partition the part of its boundary that is also contained in Γ_{int} into two sets:

$$\begin{aligned} \partial_-\kappa \;&=\; \{x \in \partial\kappa \cap \Gamma_{int} : b(x)\cdot\mu_\kappa(x) < 0\} \qquad \text{(inflow part)}\,, \\ \partial_+\kappa \;&=\; \{x \in \partial\kappa \cap \Gamma_{int} : b(x)\cdot\mu_\kappa(x) \geq 0\} \qquad \text{(outflow part)}\,. \end{aligned}$$

Given $v \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h)$, its restriction to $\bar{D} \subset \bar{\Omega}$ is denoted by $v_D = v|_{\bar{D}}$. Then, for $x \in \partial_-\kappa$ there exists a unique neighbor κ' with $x \in \partial\kappa'$ and set

$$v_\kappa^+(x) = v_\kappa\,(x)\,, \quad v_\kappa^-(x) = v_{\kappa'}\,(x)\,, \quad \lfloor v \rfloor_\kappa = v_\kappa^+ - v_\kappa^-\,.$$

Given an interior edge $e \in \mathcal{E}_{int}$, there are two elements κ_i, κ_j, with, e.g., $i > j$, that share this edge. We define

$$[v]_e = v|_{\partial\kappa_i \cap e} - v|_{\partial\kappa_j \cap e}, \quad < v >_e = \frac{1}{2}(v|_{\partial\kappa_i \cap e} + v|_{\partial\kappa_j \cap e}) \,,$$

and ν as the unit normal which points from κ_i to κ_j. We note, that μ and ν point in different directions in general and that $\lfloor \cdot \rfloor$ and $[\cdot]$ are distinct. Similarly, for $e = \partial\kappa \cap \Gamma$, we set

$$[v]_e = v|_e.$$

Finally, we introduce a discontinuity-penalization function σ defined on Γ_{int}: for a edge $e \in \mathcal{E}_{int}$, we denote the diameter of e by h_e and define

$$\sigma_e \; = \; \sigma_0 \cdot \frac{< \bar{a} p^2 >_e}{h_e} \,,$$

where $\bar{a} = ||a||$ and σ_0 is a suitably chosen positive constant.

For $u, v \in V^h$, we consider the bilinear form

$$\begin{aligned} B(u,v) \quad &= \quad \sum_{\kappa \in \mathcal{T}_h} \int_\kappa a \nabla u \cdot \nabla v dx + \sum_{\kappa \in \mathcal{T}_h} \int_\kappa (b \cdot \nabla u + cu) v dx \\ &- \quad \sum_{\kappa \in \mathcal{T}_h} \int_{\partial_- \kappa \cap \Gamma_{int}} (b \cdot \mu) \lfloor u \rfloor v^+ ds + \int_{\Gamma_{int}} \sigma [u][v] ds \\ &+ \quad \int_{\Gamma_{int}} ([u] < (a \nabla v) \cdot \nu > - < (a \nabla u) \cdot \nu > [v]) \, ds \,, \end{aligned}$$

which has been proposed in [4]. Our DG approximation of (1) is then defined as the unique $u \in V^h$ such that

$$B(u,v) = (f,v)_{L^2(\Omega)} \,, \quad v \in V^h \,. \tag{3}$$

Problem (3) can be written in matrix form as

$$Bu = f, \tag{4}$$

where we have used the same notation for a function $u \in V^h$ and the corresponding vector of degrees of freedom, and a bilinear form, e.g., $B(\cdot,\cdot)$, and its matrix representation in the space V^h. Similarly, in the following we use the same notation for functional spaces and the corresponding spaces of vectors of degrees of freedom.

We consider the following scalar product in $S_0^{\mathbf{p}}(\Omega, \mathcal{T}_h)$:

$$A(u,v) \quad = \quad \sum_{\kappa \in \mathcal{T}_h} \int_\kappa a \nabla u \cdot \nabla v dx + \int_{\Gamma_{int}} \sigma [u][v] ds \,,$$

3 Two–level overlapping preconditioners

We consider preconditioners of B of the form

$$\hat{B}^{-1} = \sum_{i=1}^{N} R_i^T B_i^{-1} R_i + R_0^T B_0^{-1} R_0,$$

where the $\{B_i\}$ are local operators associated to a partition of Ω into subdomains and B_0 is associated to a global, low–dimensional problem. More precisely, we consider a non–overlapping partition of Ω into subdomains

$$\mathcal{F}_H = \{\Omega_i\}_{1 \le i \le N},$$

of diameter $H > h$. We next extend each Ω_i to a larger region $\Omega_i' \subset \Omega$, in such a way that Ω_i' is the union of some elements in $\mathcal{T}_h$.

The first problem we need to address is the choice of the local solvers associated to the $\{\Omega_i'\}$. Here, we exploit the fact that we work with discontinuous FE functions and define our local spaces by

$$V_i = \{\, u \in V^h : \quad u_{|\kappa} = 0,\ \kappa \subset \Omega \setminus \Omega_i' \,\}, \quad 1 \le i \le N\,. \tag{5}$$

We note that a function in V_i is discontinuous and, as opposed to the case of conforming approximations, in general does not vanish on $\partial\Omega_i'$. Then, $R_i^T : V_i \to V^h$ is the natural interpolation operator from the subspace V_i into V_h and the restriction $R_i : V^h \to V_i$ puts to zero the degrees of freedom outside $\bar{\Omega}_i'$.

We showed in [6] that, in the pure hyperbolic case $a = 0$, the local operator $B_i = R_i B R_i^T : V_i \to V_i$, $1 \le i \le N$, is the approximation of a Dirichlet problem with *weakly* imposed boundary conditions on the inflow part of the boundary $\partial\Omega_i'$, which is therefore well–posed. If some diffusion is present, B_i, although having contributions from bilinear forms defined on the boundary, is not the approximation of a Dirichlet problem with weakly imposed boundary conditions on $\partial\Omega_i'$. However, it is positive–definite and the corresponding local problem in Ω_i' is well–posed.

We also note that, thanks to the choice of the local spaces, the case of zero overlap,

$$\Omega_i' = \Omega_i, \quad 1 \le i \le N,$$

can be considered, as was already noted in [3].

The first coarse solver that we consider was already introduced in [6]. It requires that the partition $\mathcal{F}_H$ is a coarse mesh $\mathcal{T}_H$. The matrix B_0 is then the approximation of our advection–diffusion problem on the *continuous, piecewise linear* FE space

$$V_0 = S^1(\Omega, \mathcal{T}_H) \cap H_0^1(\Omega) \subset V^h.$$

If $R_0^T : V_0 \to V^h$ is the natural interpolation operator from the subspace V_0 into V_h, then our coarse solver is

$$B_0 = R_0 B R_0^T,$$

and it can be easily shown to be positive–definite. In [6], we proved that this choice of coarse space leads to an optimal, scalable preconditioner for GMRES. The second coarse solver is introduced in the next section.

4 Smoothed aggregation techniques

The use of smoothed aggregation (SA) techniques allows to build coarser spaces without the need of introducing coarser triangulations for multi–level and two–level preconditioners, and is particular advantageous when dealing with problems on unstructured grids; see [7, 1, 5]. The use of such techniques also appear to be promising for DG approximations, thanks to the possibility for the smoothed coarse basis functions to 'follow' the direction of the flow b.

We first suppose that the reaction term c is identically zero. We start by associating a vector to each subdomain. Let $\tilde{\phi}_i \in V_h$ be the characteristic function of Ω_i. The functions $\{\tilde{\phi}_i\}$ span a subspace of dimension N and are good candidates for building a coarse space since they are able to reproduce constant functions. Unfortunately, they have a high energy. We note that we are working with discontinuous functions and that the term in the energy bilinear form $A(\cdot,\cdot)$ responsible for this high energy is the penalization term. Indeed, we have

$$A(\tilde{\phi}_i, \tilde{\phi}_i) \approx \frac{H^{d-1}}{h}.$$

The idea of SM techniques is then to smooth these functions out by increasing their support using the stencil of suitable polynomials in B. The property of being able to reproduce constants relies on the kernel of the operator $\mathcal{L}$.

$$\mathcal{L}1 = 0.$$

For $q \in \mathrm{N}_0$, we define

$$\phi_i = S^q \tilde{\phi}_i := (I - \omega D^{-1} B)^q \tilde{\phi}_i, \quad 1 \leq i \leq N, \quad \omega \in (0, 1],$$

where D is a suitable diagonal matrix that can be chosen, for instance, as the diagonal part of B. We note that the smoothed functions are still able to reproduce the constants.

We first consider the pure hyperbolic case $a = 0$. Given an element $\kappa \in \mathcal{T}_h$ and a neighboring element κ' that share an edge e with κ, the degrees of freedom of κ on e are only coupled with the corresponding degrees of freedom of κ' on e through the upwinding term of the bilinear form. Due to this fact:

- we need two applications of S, in order to extend the supports of the $\{\tilde{\phi}_i\}$ of one layer;
- the support only increases along the streamlines, in the **positive** direction of the flow b.

This second property appears to be extremely favorable since the exchange of information produced by the coarse solve follows the same pattern as that of the original problem.

If the diffusion is not zero, the degrees of freedom of the element κ on e are coupled to all the degrees of freedom of κ'. In this case:

- we need one application of S, in order to extend the supports of the $\{\tilde{\phi}_i\}$ of one layer and we expect the entries of the smoothed functions to be higher in the direction of the flow for convection–dominated problems;
- their support is extended in all directions.

If the reaction coefficients is not zero, constant functions are not in general reproduced, but we expect this to be balanced by the better conditioning of B.

Our coarse space is then defined as

$$V_0 := \text{span}\,\{\phi_i\}.$$

We remark that a SA technique provides now *all* the components of our preconditioner:

- the coarse space, through the matrix R_0^T, the columns of which are the vectors $\{\phi_i\}$;
- the local solvers, since the overlapping subdomains can be chosen as the supports of the functions $\{\phi_i\}$.

For the last property, we remark that, in the diffusive case the overlap between the subdomains is $O(qh)$.

5 Numerical results

In this section, we show some test cases for two simple problems in two dimensions. They are for uniform meshes on the unit square, consisting of $2(n \times n)$ triangles, and linear finite elements. We impose Dirichlet conditions weakly. We have employed $GMRES$ without restart. We have stopped our iterations once a reduction of the residual norm of 10^{-6} is achieved or after 100 iterations. We note that, for coarse spaces built with SA techniques, the overlap $\delta = qh$ is determined by the degree q of the smoothing operator.

We first consider the Poisson equation with inhomogeneous Dirichlet conditions:

$$-\Delta u = xe^y \quad \text{in } \Omega\,, \quad u = -xe^y \quad \text{on } \Gamma\,.$$

and partitions into $nc \times nc$ squares ($H = 1/nc$), with $nc = 2, 4, 8, 16, 32$. Table 1 shows the iteration counts for the two algorithms, as functions of h and the inverse of the relative overlap. We have also considered the case of zero overlap, denoted by $H/\delta = \infty$.

Both methods appear to be rather insensitive to the size of the original problem and the number of subdomains, when the relative overlap δ/H is positive

two level (I)						
			H/δ			
n	nc	∞	16	8	4	2
16	4	13	-	-	13	14
32	4	15	-	13	12	13
32	8	13	-	-	13	15
64	4	19	15	14	13	13
64	8	16	-	13	13	14
64	16	13	-	-	13	15
128	4	25	18	16	14	13
128	8	35	15	14	13	14
128	16	15	-	13	13	15
128	32	12	-	-	13	15

two level (II), $\omega = 2/3$						
			H/δ			
n	nc	∞	16	8	4	2
16	4	21	-	-	22	18
32	4	28	-	30	25	20
32	8	25	-	-	26	22
64	4	37	40	33	27	21
64	8	33	-	35	31	25
64	16	26	-	-	27	25
128	4	48	53	44	35	28
128	8	43	47	42	34	27
128	16	34	-	37	35	28
128	32	26	-	-	27	25

Figure 1: Poisson problem with standard coarse space (left) and a coarse space built using a smooth aggregation technique (right).

and fixed. The two algorithms appear to be optimal and scalable. We also note that the iteration numbers decrease when the relative overlap increases, and that the iterations for method (II) (smoothed aggregation) are roughly double those of method (I) (standard coarse space). We remark that for symmetric positive–definite problems, the condition number grows linearly with H/δ for method (I), while we expect a quadratic growth for method (II); see [5].

The case of zero overlap requires a special discussion. Our results show that the number of iterations obtained are generally comparable to, but slightly higher than, those obtained in the case of $\delta > 0$ for both algorithms. From our numerical results for algorithm (I), we are unable to deduce whether it is optimal or non–optimal, with the number of iterations growing as a power of H/h. We refer to [3] for a method with the same local solvers but a different coarse space, which exhibits a rate of convergence that appears to grow linearly with H/h. On the other hand, we note that for algorithm (II), comparable numbers of iterations are obtained when the ratio H/h is fixed.

However, we believe that due to the minimal communication between the subdomains and the relatively small iteration counts that we have obtained, the algorithms with zero overlap might be competitive in practice.

We next consider the advection-diffusion equation

$$-\Delta u + b \cdot \nabla u + cu = f \quad \text{in } \Omega\,, \quad u = 0 \quad \text{on } \Gamma\,,$$

with constant coefficients and weakly–imposed zero Dirichlet boundary conditions. We consider the case

$$b = -(3\pi, 3\pi), \quad c = 3\pi^2.$$

The right-hand side f is always chosen such that the exact solution is $u = xe^{xy}\sin(\pi x)\sin(\pi y)$. The numbers of iterations are shown in Table 2.

two level (I): $b = -(3\pi, 3\pi)$, $c = 3\pi^2$						
			H/δ			
n	nc	∞	16	8	4	2
16	4	14	-	-	14	15
32	4	16	-	14	13	14
32	8	12	-	-	13	15
64	4	19	15	14	13	14
64	8	13	-	13	13	15
64	16	10	-	-	12	15
128	4	24	19	16	14	14
128	8	35	13	13	13	15
128	16	11	-	11	11	15

two level (II), $\omega = 2/3$						
			H/δ			
n	nc	∞	16	8	4	2
16	4	22	-	-	17	15
32	4	28	-	24	20	16
32	8	28	-	-	22	19
64	4	38	33	27	22	16
64	8	39	-	32	28	21
64	16	31	-	-	25	23
128	4	50	45	37	30	23
128	8	53	45	40	32	23
128	16	43	-	37	34	26

Figure 2: Advection–diffusion problem with standard coarse space (left) and a coarse space built using a smooth aggregation technique (right).

As for the Poisson problem with non–vanishing overlap, the iteration counts decrease when the overlap increases and are independent of the number of subdomains and the problem size. SA techniques also give numbers of iterations that are double those for a standard coarse space.

The behavior for zero overlap appears to be more regular when a standard coarse space is employed. The number of iterations appears to grow like H/h, *when h is fixed*. For a fixed value of H/h, slower convergence rates are obtained for h larger. We can then conclude that, for the case of zero overlap, the iteration counts are indeed bounded by a $C(H/h)$, with C a suitable constant; see also [3].

On the other hand, with a SA technique comparable numbers of iterations are obtained for fixed h, regardless the value of H, but the method does not appear to be optimal in this case.

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