We conclude with four observations:

(a) The curves $z(s)$ and $z(s, t)$ were not required to be simple curves. This made the task of approximating them with continuously differentiable curves very easy.

(b) The Cauchy integral theorem also holds if the function $z(s)$ describing the curve $C$ is merely piecewise continuously differentiable. For in this case we carry out the integration by parts in (6) over each subinterval on which $z_s(s)$ is continuous. It follows from the formula on the right side of (6) that $z_s^k(s)$ tends to $z_s(s)$ uniformly on every subinterval of [0,1] on which $z_s(s)$ is continuous. Around points of discontinuity of $z_s(s)$, the functions $z_s^k(s)$ remain uniformly bounded. This is sufficient to conclude that (7) holds as $k$ tends to $\infty$.

(c) Suppose that the closed curve $C$ is homologous to zero in $D$. Then it can be decomposed as a finite sum of closed curves $C_n$ each of which is homotopic in $D$ to a point. If $C$ is continuously differentiable, each curve $C_n$ is piecewise continuously differentiable, so by what was proved in (b), the integral of $f$ over each curve $C_n$ is zero. But then so is their sum. This shows that the Cauchy integral theorem holds for curves homologous to zero relative to $D$.

(d) The Cauchy integral theorem holds for rectifiable curves, as is easily derived from the result in this note. I don't know of any application of the Cauchy theorem where we need to go beyond piecewise continuously differentiable curves.

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# Taming a Hydra of Singularities

**Folkmar Bornemann and Thomas Schmelzer**

**1. INTRODUCTION.** We invite the reader to contemplate for one moment the following remarkable limit problem (where $g^{[n]} = g \circ g \circ \cdots \circ g$ denotes the $n$-fold iteration of $g$):

*Consider a function $f : \mathbb{R} \to \mathbb{R}$ that is bounded and continuous. Does the sequence of violently oscillatory integrals*

$$\frac{1}{\pi}\int_0^{\pi} f(\tan^{[n]} x)\,dx \qquad (n = 1, 2, 3, \dots)$$

*have a limit as n approaches infinity? If so, what is the limit?*

Figure 1 gives just a faint impression of the Herculean drama caused by the iterated function $\tan^{[n]} x$. Starting with the singularities of $\tan x$ at $x = (k + 1/2)\pi$ $(k \in \mathbb{Z})$ each singularity of the $n$-fold iteration $\tan^{[n]} x$ breeds, like the mythic Hydra, countably many new singularities for the next iteration $\tan^{[n+1]} x$, which accumulate in their respective ancestor. As $x$ moves from one singularity to the next, the values of $\tan^{[n]} x$

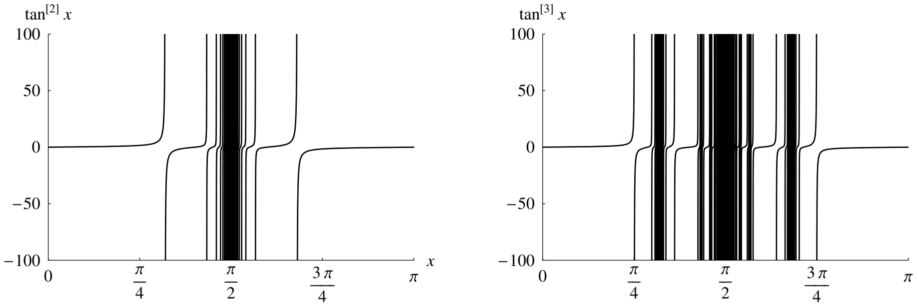


**Figure 1.** Graph of $\tan^{[2]} x$ (left) and $\tan^{[3]} x$ (right).

go all the way from $-\infty$ to $+\infty$. Asymptotically, as $n \to \infty$, these proliferating singularities become *dense* in $[0, \pi]$.

So, having this desolate picture in mind, what would possibly be a reasonable guess about the limit of the integrals? If we were more gentle and put tanh in place of tan, the problem would allow for an easy solution. In fact, we would then have the inequality $0 < \tanh x < x$ and therefore, by the monotonicity of tanh, the monotone limit

$$\lim_{n\to\infty} \tanh^{[n]} x = 0 \qquad (x > 0). \tag{1}$$

(Note that the contraction mapping theorem does not apply here since the fixed point at $x = 0$ is *parabolic*; that is, the derivative of tanh at $x = 0$ is one.) Hence, by the dominated convergence theorem, using the boundedness and continuity of $f$, we would obtain asymptotic focussing on the value of $f$ at zero:

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^\pi f(\tanh^{[n]} x)\,dx = \frac{1}{\pi}\int_0^\pi f(\lim_{n\to\infty} \tanh^{[n]} x)\,dx = \frac{1}{\pi}\int_0^\pi f(0)\,dx = f(0).$$

Quite to the contrary, we have $\lim_{n\to\infty} \tan^{[n]} x = 0$ for just countably many $x$, that is, for a negligible set of measure zero. Generically in $x$, the sequence $\tan^{[n]} x$ oscillates widly as $n \to \infty$, thereby taking values that are quite randomly spread in $(-\infty, \infty)$. One might thus expect that the integrals would, if at all, converge to some averaged value of $f$.

However, in this note we prove, by taking advantage of unexpectedly explicit evaluations of the integrals involved, that, quite remarkably, the limit focusses in exactly the same way on the value of $f$ at zero:

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^\pi f(\tan^{[n]} x)\,dx = f(0). \tag{2}$$

To begin, we briefly explain why the integrals of the sequence exist. Because $\tan^{[n]} x$ is defined almost everywhere on $[0, \pi]$ (with the exception of the countably many singularities already discussed), the integrand $f(\tan^{[n]} x)$ is bounded and continuous almost everywhere. Therefore, it is (Riemann) integrable.

**2. A FIRST PROOF: POTENTIAL THEORY.** We set the scene with some heavier theoretical machinery; an elementary proof that does not rely on complex analysis will be given later. Throughout this note we assume that $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous. Now, from potential theory (see [**2**, Theorems 15.1a, 15.4d]) we know that there is a function $F(z)$ holomorphic in the upper complex half-plane $H = \{z : \operatorname{Im} z > 0\}$ such that the harmonic function $\operatorname{Re} F(z)$ is bounded and has boundary values given by $f$, that is,

$$\operatorname{Re} F(x + iy) \to f(x) \qquad (x \in \mathbb{R}) \tag{3}$$

as the real number $y$ approaches zero from above. The holomorphic function $F$ is *unique* up to a purely imaginary additive constant. For the sake of simplicity, we further *assume that $F$ itself, not just* $\operatorname{Re} F$, *is bounded.* This additional assumption will be dropped in the elementary, real analysis proof of section 4, which will also show that the identities (6) and (8) remain valid then.

The substitution $\xi = \tan(x)$ transforms the integral as follows:

$$\frac{1}{\pi}\int_0^{\pi} f(\tan^{[n+1]} x)\, dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\tan^{[n]} \xi)}{1+\xi^2}\, d\xi. \tag{4}$$

In view of (3), the boundedness of $F$, and the fact that the correspondence $z \mapsto \tan z$ maps $H$ into itself, we can appeal to the dominated convergence theorem to conclude that

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi} f(\tan^{[n+1]} x)\, dx &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\tan^{[n]} \xi)}{1+\xi^2}\, d\xi \\ &= \lim_{\epsilon\downarrow 0} \operatorname{Re} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{F(\tan^{[n]}(\xi + i\epsilon))}{1+(\xi+i\epsilon)^2}\, d\xi. \end{aligned}\tag{5}$$

Fix $\epsilon$ in (0, 1). If $R > 1$ we denote by $\mathcal{C}_R$ the half-circle with center $i\epsilon$ that connects $-R + i\epsilon$ through $i(R+\epsilon)$ with $R + i\epsilon$. Cauchy's residue theorem yields, for the encircled pole $z = i$ of $1/(1+z^2)$ with residue $-i/2$,

$$\frac{1}{\pi}\int_{-R}^{R} \frac{F(\tan^{[n]}(\xi + i\epsilon))}{1+(\xi+i\epsilon)^2}\, d\xi = 2\pi i \cdot \frac{1}{2i}\cdot\frac{1}{\pi} F(\tan^{[n]} i) + \frac{1}{\pi}\int_{C_R} \frac{F(\tan^{[n]} z)}{1+z^2}\, dz.$$

Since $|F|$ is bounded (say, $|F(z)| \le M$ for $z$ in $H$) the latter integral can be estimated by

$$\left|\frac{1}{\pi}\int_{C_R} \frac{F(\tan^{[n]} z)}{1+z^2}\, dz\right| \le \frac{M}{\pi}\int_{C_R} \frac{d|z|}{|1+z^2|} \sim \frac{M}{\pi}\cdot\frac{\pi\cdot R}{R^2} = M\, R^{-1}$$

as $R \to \infty$. Hence, by letting $R \to \infty$ and observing that $\tan^{[n]} i = i \tanh^{[n]} 1$ we obtain

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{F(\tan^{[n]}(\xi + i\epsilon))}{1+(\xi+i\epsilon)^2}\, d\xi = F(i \tanh^{[n]} 1).$$

Combined with (5) this leads to the simple expression

$$\frac{1}{\pi}\int_0^{\pi} f(\tan^{[n+1]} x)\, dx = \operatorname{Re} F(i \tanh^{[n]} 1) \to f(0) \tag{6}$$

as $n \to \infty$, which finally proves our claim (2). Here, the asserted limit follows from the boundary representation (3) of $f$ and from $\tanh^{[n]} 1 \downarrow 0$ (see (1)).

**3. INTERMEZZO: EVALUATION OF THE INTEGRALS.** The expression (6) is a convenient tool for calculating relevant integrals explicitly. An example is given by $f(t) = \sin^2 t$. Here, we have

$$F(z) = -ie^{iz}\sin z,$$

which is obviously bounded and holomorphic in $H$ and satisfies $\operatorname{Re} F(t) = \sin^2 t$ for real $t$. In this way we get

$$\frac{1}{\pi}\int_0^\pi \sin^2(\tan^{[n+1]} x)\,dx = e^{-\tanh^{[n]} 1}\,\sinh(\tanh^{[n]} 1).$$

In particular

$$\frac{1}{\pi}\int_0^\pi \sin^2(\tan\tan x)\,dx = e^{-\tanh 1}\,\sinh(\tanh 1) = 0.39099\,21621\,51530\ldots. \tag{7}$$

Generally, there is no point in guessing $F$. However, the bounded harmonic function $\operatorname{Re} F$ is *uniquely* given by the Poisson integral of $f$ (see [**2**, (15.4-3)]), which is essentially just the real part of Cauchy's integral of $F$ taken along the real axis,

$$\operatorname{Re} F(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\,f(t)}{y^2+(x-t)^2}\,dt \qquad (y > 0).$$

Hence, using (6), we obtain the real and computionally very useful formula

$$\frac{1}{\pi}\int_0^\pi f(\tan^{[n+1]} x)\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\epsilon_n\,f(t)}{\epsilon_n^2+t^2}\,dt \qquad (\epsilon_n = \tanh^{[n]} 1). \tag{8}$$

As a second example, we have used this formula for $f(t) = \cos\log(1+t^2)$ to get

$$\begin{aligned}\frac{1}{\pi}\int_0^\pi \cos\log(1+(\tan\tan x)^2)\,dx &= \frac{\tanh(1)}{\pi}\int_{-\infty}^{\infty} \frac{\cos\log(1+t^2)}{\tanh^2(1)+t^2}\,dt \\ &= 0.54972\,79152\,95795\ldots.\end{aligned}$$

The fast and accurate numerical evaluation of the latter, infinite range oscillatory integral can be based on the methods described in [**1**, chap. 1].

**4. AN ELEMENTARY SECOND PROOF.** The real formula (8) can be written in the form

$$\frac{1}{\pi}\int_0^\pi f(\tan^{[n+1]} x)\,dx = \int_{-\infty}^{\infty} f(t)\,\phi_{\epsilon_n}(t)\,dt \qquad (\epsilon_n = \tanh^{[n]} 1), \tag{9}$$

where we have introduced, for short, the density $\phi_\epsilon$ of the Cauchy distribution,

$$\phi_\epsilon(x) = \frac{1}{\pi}\frac{\epsilon}{\epsilon^2+x^2}.$$

Before we give an independent, elementary real analysis proof of this formula, we use it to establish our main assertion (2) once more. All we have to show here is the well-known fact [**5**, Theorem 9.9] that $\phi_\epsilon$ is an approximate $\delta$-function; that is,

$$\int_{-\infty}^{\infty} f(t)\,\phi_\epsilon(t)\,dt \to f(0)$$

as $\epsilon$ approaches 0 from above. In fact, the change of variables $t = \epsilon\tau$ gives, by appealing to the dominated convergence theorem and using the boundedness and continuity of $f$,

$$\int_{-\infty}^{\infty} f(t)\,\phi_\epsilon(t)\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\epsilon\tau)}{1+\tau^2}\,d\tau \;\to\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(0)}{1+\tau^2}\,d\tau = f(0).$$

To complete the proof, we derive (9) from (4), which we conveniently rewrite in terms of the function $\phi_\epsilon$ as

$$\frac{1}{\pi}\int_0^{\pi} f(\tan^{[n+1]} x)\,dx = \int_{-\infty}^{\infty} f(\tan^{[n]}\xi)\,\phi_1(\xi)\,d\xi.$$

We do so by establishing a kind of integral *symmetry law* between the twins tan and tanh,

$$\int_{-\infty}^{\infty} f(\tan^{[n]}\xi)\,\phi_\epsilon(\xi)\,d\xi = \int_{-\infty}^{\infty} f(t)\,\phi_{\tanh^{[n]}\epsilon}(t)\,dt \qquad (\epsilon > 0).$$

By induction, it suffices to prove this for $n = 1$. To make that work we relax the assumptions on $f$: it is still bounded, but continuous just *almost everywhere*. (In fact, $f$ bounded and measurable would be sufficient. The limit (2) then holds if $f$ is at least continuous at zero.) To this end, we transform the integrals (using the principal branch of arctan) as follows, exploiting absolute convergence to interchange the order of summation and integration:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(\tan\xi)\,\phi_\epsilon(\xi)\,d\xi &= \sum_{k=-\infty}^{\infty}\int_{(k-1/2)\pi}^{(k+1/2)\pi} f(\tan\xi)\,\phi_\epsilon(\xi)\,d\xi \\
&= \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\eta)\,\phi_\epsilon(k\pi + \arctan\eta)}{1+\eta^2}\,d\eta \\
&= \int_{-\infty}^{\infty} f(\eta)\left(\frac{1}{1+\eta^2}\sum_{k=-\infty}^{\infty}\phi_\epsilon(k\pi + \arctan\eta)\right)d\eta \\
&= \int_{-\infty}^{\infty} f(\eta)\,\phi_{\tanh\epsilon}(\eta)\,d\eta.
\end{aligned}$$

The evaluation of the infinite series that yields the last identity can be done either by looking it up in a table such as [**4**, (5.1.25.3)] or by simply typing

$$\texttt{FullSimplify}\left[\frac{1}{1+\eta^2}\sum_{k=-\infty}^{\infty}\frac{\epsilon}{\epsilon^2 + (k\pi + \arctan(\eta))^2},\ \{\epsilon > 0, \eta \in \mathbb{R}\}\right]$$

into *Mathematica*, which gives $\pi \cdot \phi_{\tanh\epsilon}(\eta)$ right away in the equivalent form

$$\frac{1}{\coth(\epsilon)\eta^2 + \tanh(\epsilon)}.$$

**Remark.** Admittedly, these two suggestions for evaluating the infinite series might, for some readers, not qualify as *proof*. For the sceptical reader we recommend the exercise of evaluating the infinite series as a partial fraction expansion. In fact, by analogy with the elementary, real analysis derivation of $\sum_{k=-\infty}^{\infty} 1/(k\pi + x)^2 = 1/\sin^2 x$ presented in [**3**, p. 198], we get

$$\sum_{k=-\infty}^{\infty} \frac{\epsilon}{\epsilon^2 + (k\pi + x)^2} \leftarrow \frac{1}{n} \sum_{k=-n/2}^{n/2-1} \frac{\sinh(2\epsilon/n)/2}{\sinh^2(\epsilon/n) + \sin^2((k\pi + x)/n)}$$
$$= \frac{\sinh(2\epsilon)/2}{\sinh^2 \epsilon + \sin^2 x}$$

as $n = 2^\nu \to \infty$. Here, the identity follows recursively from the simple case $\nu = 1$. A little endurance in massaging the hyperbolic and trigonometric functions helps us to conclude the proof with the identity

$$\frac{1}{1+\eta^2} \cdot \frac{\sinh(2\epsilon)/2}{\sinh^2 \epsilon + \sin^2(\arctan \eta)} = \frac{\tanh \epsilon}{\tanh^2 \epsilon + \eta^2}.$$

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# Pick's Theorem via Minkowski's Theorem

**M. Ram Murty and Nithum Thain**

In 1899, Georg Alexander Pick published one of his most beautiful theorems [**12**]. This theorem provided a formula for easily calculating the area of a planar polygon $P$ whose vertices have integer coordinates. Such a polygon is called a *lattice polygon*, since the