Abstract. The affine morphological scale space (AMSS), a nonlinear diffusion equation, is uniquely characterized by a set of desirable axioms suggested for multiscale smoothing of images. Its numerical approximation, however, poses considerable problems: monotone finite-difference schemes require unbounded families of stencils and Moisan’s (1998) morphological scheme has unbounded structuring elements. In the current paper we show that the convergence of the scheme still holds if we suitably localize its structuring elements. Further, by a localization to $3 \times 3$ blocks of pixels, we will construct a new family of image operators, one of which has denoising capabilities that are superior among the competing image operators, including the classical median filter.

1. Introduction. Natural images depict an inherently multiscaled world; we experience, by getting closer or zooming-in, that the visual aspect of some objects changes and new structures arise (convincingly illustrated in Guichard and Morel 1997, Fig. 10). A multiscale analysis, or scale space, of images tries to emulate this physical and perceptual fact by generating a series of smoothings of a digital image at different scales: at a given scale the finer-scale structures of the image are removed but the coarser-scale structures are just minimally modified.

A gray-tone image is modeled by a bounded real function $u$ defined on $\mathbb{R}^2$; the value $u(x)$ represents the luminance at the point $x$. A scale space is therefore given by a family of operators $T_t$, where $T_t u$ represents the processing of the image $u$ at the scale $t$, fixing $T_0 u = u$ at the scale $t = 0$. In their seminal paper Alvarez et al. (1993) have shown that a set of natural demands on the scale space, called axioms, make $T_t$ the evolution of a certain class of nonlinear diffusion equations. We follow the presentation in Guichard and Morel (1997, §2) and consider the family $(\text{CAU})$ of causality axioms

(PYR) pyramidality: the scale space embeds as $T_t = T_{t,0}$ into a two parameter family $T_{t,s}$ of operators satisfying $T_{t+s} = T_{t+s,s} \circ T_{t,s}$, $s, t \geq 0$;

(COM) comparison principle: if $u(y) < v(y)$ for $y \neq x$ in a vicinity of the point $x$, then $T_{t+h} u(x) \leq T_{t+h} v(x)$ for $h > 0$ sufficiently small;

(LOC) locality: there exists a function $F(A, p, x, c, t)$, continuous in $A$, such that, if $u$ is a quadratic form, $T_{t+h} u(x) = u(x) + F(D^2 u(x), Du(x), x, u(x), t) h + o(h)$;

the family $(\text{MORPH})$ of morphology axioms

(CON) contrast invariance: for any nondecreasing real function $g$, operating on images $u$ by $g(u)(x) = g(u(x))$, there is $T_{t,s} \circ g = g \circ T_{t,s}$;

(TRS) translation invariance: for the translation operators $\tau_x$, operating on images $u$ by $\tau_x(u)(x_0) = u(x_0 - x)$, there is $T_{t,s} \circ \tau_x = \tau_x \circ T_{t,s}$;

extended by the axiom

(NEG) negative invariance: there is $T_{t,s} u = -T_{t,s}(-u)$;

and a geometric invariance axiom

(AFF) affine invariance: for $\phi \in \text{GL}(\mathbb{R}^2)$, operating on images $u$ by $\phi(u)(x) = u(\phi(x))$, there is $T_{t,s} \circ \phi = \phi \circ T_{t,s}$ with $\lambda = \sqrt{\det \phi}$.

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Particular cases of the affine invariance, called *isotropy* \((\text{ISO})\) and *zoom invariance* \((\text{ZOOM})\), are obtained if we restrict the matrix \(A\) to \(A \in \text{O}(\mathbb{R}^2)\) or \(A = \lambda I, \lambda > 0\), respectively.

Now, Alvarez et al.’s (1993, Thm. 5) fundamental result shows that a scale space satisfying the axioms \((\text{CAU})\), \((\text{MORPH})\), \((\text{NEG})\), \((\text{ISO})\), and \((\text{ZOOM})\) is the evolution associated with the unique viscosity solutions of a nonlinear partial differential equation of the form

\[
u_t = |Du| \beta(t \text{curv} u), \quad \text{curv} u = \text{div} \frac{Du}{|Du|},
\]

where \(\beta\) is a continuous odd real function; see Guichard and Morel (1997, Thm. 8). Note that \(\text{curv} u(x)\) is the curvature of the level set of \(u\) passing through \(x\); so the equation has a geometric interpretation. The simplest instance of such an equation is given, up to a nonlinear rescaling of \(t\), by the *mean curvature motion*

\[
u_t = |Du| \text{curv} u.
\]

For \((\text{ISO})\) and \((\text{ZOOM})\) replaced by the full affine invariance \((\text{AFF})\) Alvarez et al. (1993, Thm. 5) have shown that necessarily \(\beta(s) = cs^{1/3}, c > 0\). In this case one gets, uniquely up to a rescaling, the *affine morphological scale space*

\[
u_t = |Du| (\text{curv} u)^{1/3}.
\]

This geometric differential equation was independently derived by Sapiro and Tannenbaum (1994), who considered the affine invariant smoothing of curves.

A few numerical methods for the approximation of the mean curvature motion and the AMSS have been suggested in the literature, for a survey see Cao (2003, §5.2). We focus on some aspects that are relevant for this paper.

Alvarez and Morel (1994, §10) have developed finite-difference methods based on a 9-point stencil and with coefficients that optimize the non-monotonicity of the scheme. These schemes are consistent, but nothing is known of their convergence.

On the other hand, for the mean curvature motion, the monotone finite-difference scheme of Crandall and Lions (1996) has the unpleasant feature that the stencil size becomes unbounded in the limit that yields convergence. In fact, these authors show the necessity of such a behavior of monotone schemes. Finally, Guichard and Morel (2000, Fig. 24.10) have demonstrated by numerical experiments that the 9-point stencil finite-difference scheme for the AMSS, if not applied on a subpixel resolution, is of limited usability for the denoising of digital images.

Morphological and affine invariant geometric schemes for the evolution of level sets have been developed, for the AMSS, by Moisan (1998) and, for more general curvature equations, by Cao and Moisan (2001) and Cao (2003, Chap. 6). They are fairly expensive if applied to the denoising of digital images: 8-bit gray-tone images have up to 256 different level sets whose many different components have to be approximated on a subpixel resolution.

There are successful morphological image operators defined on a pixel resolution, such as the median filter. Guichard and Morel (1997, 2000) have developed a general theory that the iteration of these filters, if suitably generalized and rescaled, converges to a curvature equation. In particular, they have shown that this way the median filter is related to the mean curvature motion. On the other hand, by the work of Pasaquin (1999), we know that all the curvature equations of the form (1), in particular the AMSS, can be obtained by such an iterative filtering.
The Contribution of Our Paper and Its Outline. By Matheron’s (1975) general theory, morphological operators \( T \) can be represented as sup-inf operators, 

\[
Tu(x) = \sup_{B \in B} \inf_{y \in B} u(x + y),
\]

generated by a suitable set \( B \) of structuring elements. In §2 we introduce a convenient partial order of families of structuring elements that will help in comparing them later on. In §3 we make the family of structuring elements explicit for the morphological operators underlying the propagator of Moisan’s (1998) geometrical scheme for the AMSS. This family of structuring elements turns out to be unbounded, which partly explains the difficulties in its theory and numerical realization. We then aim, from two different point of views, at localizing the structuring elements. On the one hand, in §4, we study the question whether suitably restricting the structuring elements to a disk suffices to still prove convergence. On the other hand, in §5, we study the question whether a discretization of the structuring elements on a \( 3 \times 3 \) block of pixels generates new interesting image operators that favorably compete with the denoising capabilities of the iterated median filter. As it turns out, we can give positive answers to both questions.

2. Sup-Inf Operators and Families of Structuring Elements. We study morphological operators, that is, operators satisfying the axioms (CON) and (TRS). They operate on some suitable class of functions \( u : \mathbb{R}^2 \to \mathbb{R} \), such as the (semi-)continuous functions or the ones that are piecewise constant on pixels. These operators will generally be given in their Matheron representation (2) as sup-inf operators generated by a set \( B \), which is a subset of the powerset of \( \mathbb{R}^2 \), called the family of structuring elements. The dual, or negative, operator \( T^* \) of \( T \) is defined as

\[
T^*u(x) = -T(-u)(x) = \inf_{B \in B} \sup_{y \in B} u(x + y).
\]

Note that self dual operators, \( T = T^* \), satisfy the axiom (NEG) of negative invariance.

We will frequently have to compare different families of structuring elements. A convenient way of dealing with that is by introducing a partial order of these families. We define that one family \( B_0 \) is dominated by another one \( B_1 \), denoted by \( B_0 \preceq B_1 \), if for each \( B \in B_0 \) there is a \( B' \subset B \) with \( B' \in B_1 \). Note that \( B_0 \subset B_1 \) already implies \( B_0 \preceq B_1 \). Now, the catch is that dominance induces the corresponding inequality between the generated sup-inf operators,

\[
B_0 \preceq B_1 \Rightarrow \sup_{B \in B_0} \inf_{y \in B} u(x + y) \leq \sup_{B \in B_1} \inf_{y \in B} u(x + y).
\]

The simple one-line argument is that \( B_0 \preceq B_1 \) yields

\[
\sup_{B \in B_0} \inf_{y \in B} u(x + y) \leq \sup_{B \in B_0} \inf_{y \in B'} u(x + y) \leq \sup_{B \in B_1} \inf_{y \in B} u(x + y),
\]

where the first inequality follows from \( B' \subset B \), and the second one from \( B' \subset B_1 \). We say that two families \( B_0 \) and \( B_1 \) are of equal strength, notated as \( B_0 \asymp B_1 \), if \( B_0 \preceq B_1 \) and \( B_1 \preceq B_0 \). Thus, families of equal strength generate the same sup-inf and inf-sup operators.

If the family \( B \) of structuring elements is a finite set there is the well-defined subfamily \( \bar{B} \) of elements of \( B \) that are minimal with respect to set inclusion,

\[
\bar{B} = \{ B \in B : B' \subset B \land B' \in B \Rightarrow B = B' \} \times B.
\]

We call this smallest subfamily \( \bar{B} \) of equal strength the basis of the finite family \( B \).
3. The Morphological Scheme of Moisan. We recall the basic definitions and results of Moisan’s (1998) work on an affine invariant morphological scheme for the AMSS. (See also Cao and Moisan (2001), and Cao (2003, §6.1).)

- Let \( C \) be an oriented simple curve. A chord is a line segment of the form \( C(s), C(t) \) that does not intersect \( C \) for any point with parameter between \( s \) and \( t \). The induced chord set \( C_{s,t} \) is the connected closed set enclosed by the chord \( C(s), C(t) \), its endpoints and the curve \( C([s,t]) \); see Fig. 1. The boundary of a chord set gets the orientation induced by \( C \); it has therefore an oriented area \( \sigma \) and we call \( C_{s,t} \) a \( \sigma \)-chord set of \( C \). The set of all \( \sigma \)-chord sets of \( C \) is denoted by \( K_C \).

- An open subset \( S \) of the plane \( \mathbb{R}^2 \) is called a contour set, short C-set, if it has a finite number of connected components and the boundary of any connected component, called a component of \( \partial S \), is a finite disjoint union of semiclosed piecewise convex oriented curves. The (positive) orientation is chosen such that \( S \) lies to the left of its boundary. Note that the boundary of a C-set admits a tangent almost everywhere and that finite unions of convex sets are C-sets. The set of C-sets will be denoted by \( \mathcal{C}(\mathbb{R}^2) \).

Now, following Moisan (1998, Def. 2), the \( \sigma \)-affine erosion\(^1 \) \( E_\sigma S \) of a C-set \( S \) is obtained by removing from \( S \) all \( \sigma \)-chord sets of the components of \( \partial S \) (see Fig. 1),

\[
E_\sigma S = S \setminus \bigcup_{K \in K_C : 0 \leq \sigma' < \sigma} K.
\]

(4)

The result is again a C-set, see Moisan (1998, Prop. 1). The erosion operator can be extended to l.s.c. functions \( u : \mathbb{R}^2 \to \mathbb{R} \) by two well-defined extension steps (see Moisan 1998, p. 416): first, by extending to arbitrary open sets \( U \subset \mathbb{R}^2 \) via

\[
E_\sigma U = \bigcup_{S \in \mathcal{C}(\mathbb{R}^2) : S \subset U} E_\sigma S,
\]

(5)

\(^1\)As already noted by Moisan (1998, p. 413) and Guichard and Morel (2000, Rem. 13.5), the denomination “erosion” for the operator \( E_\sigma \) does not correspond to the standard terminology used in mathematical morphology. There, an erosion is generally defined as an operator \( E \) between complete lattices satisfying \( E(\bigwedge X_i) = \bigwedge E(X_i) \), see Heijmans (1994, Def. 3.12). However, we follow the usage for the affine morphological scale space and will maintain the name “erosion” (and, dually, “dilation”) for the operator \( E_\sigma \) and its localizations discussed below.
and second by applying the set operator to the level sets of \( u \) via Matheron’s threshold decomposition principle (see Cao 2003, Prop. 4.12)

\[
E_{\sigma}u : x \mapsto \sup \{ \lambda \in \mathbb{R} : x \in E_{\sigma,\lambda}(u) \}, \quad \chi_{\lambda}(u) = \{ x : u(x) > \lambda \}.
\]

Moisan (1998, Prop. 6) has proven that the resulting operator is affine invariant and morphological, that is, satisfies the axioms (AFF) and (MORTH). Because of the last property, a theorem of Matheron (see Cao 2003, Thm. 4.19) implies that \( E_{\sigma} \) can be written as a sup-inf operator

\[
E_{\sigma}u(x) = \sup_{B \in \mathcal{B}_{\sigma}} \inf_{y \in B} u(x + y).
\]  

(6)

In fact, the set \( \mathcal{B}_{\sigma} \) of structuring elements can be made fairly explicit.

**Theorem 1.** The \( \sigma \)-affine erosion operator \( E_{\sigma} \) has a Matheron representation (6) with the set of structuring elements chosen as \( \mathcal{B}_{\sigma}^{\text{aff}} \sim \mathcal{B}_{\sigma} \),

\[
\mathcal{B}_{\sigma}^{\text{aff}} = \{ B \subset \mathcal{C}(\mathbb{R}^2) : 0 \in B, \text{each } K \in \mathcal{K}_{\sigma'}(\partial B) \text{ with } 0 \text{ on its chord satisfies } \sigma' > \sigma \},
\]

which is affine invariant in the sense that \( \phi(\mathcal{B}_{\sigma}^{\text{aff}}) = \mathcal{B}_{\sigma}^{\text{aff}} \) for any affine map \( \phi \).

**Proof.** As stated by Moisan (1998, p. 416), the general theory of Matheron representations (see Guichard and Morel (2000, Chap. 8) or Heijmans (1994, §5.4)) yields

\[
E_{\sigma}u(x) = \sup_{B \in \mathcal{B}} \inf_{y \in \sqrt{\sigma}B} u(x + y), \quad \mathcal{B} = \{ B \subset \mathbb{R}^2 : B \text{ open and } 0 \in E_1(B) \}.
\]

However, the proof of this fact is fairly straightforward and could have been given without going back to that theory.

Since the property of being a chord set with 0 on its chord is left invariant under affine maps \( \phi \) and since area generally scales with \( |\det \phi| \) we immediately get the asserted invariance property. The particular case of the map \( \phi = \sqrt{\sigma}I \) gives

\[
\mathcal{B}_{\sigma}^{\text{aff}} = \{ \sqrt{\sigma}B : B \in \mathcal{B}_{1}^{\text{aff}} \}.
\]  

(7)

Below, we will prove that \( \mathcal{B}_{1}^{\text{aff}} \sim \mathcal{B} \) which by (3) yields the asserted representation

\[
E_{\sigma}(u)(x) = \sup_{B \in \mathcal{B}} \inf_{y \in \sqrt{\sigma}B} u(x + y) = \sup_{B \in \mathcal{B}_{1}^{\text{aff}}} \inf_{y \in \sqrt{\sigma}B} u(x + y) = \sup_{B \in \mathcal{B}^{\text{aff}}} \inf_{y \in \mathcal{B}} u(x + y).
\]

We prove \( \mathcal{B}_{1}^{\text{aff}} \sim \mathcal{B} \) by showing \( \mathcal{B}_{1}^{\text{aff}} \subset \mathcal{B} \), namely, for \( B \in \mathcal{B}_{1}^{\text{aff}} \) is \( 0 \in E_1(B) \). Assuming the contrary, the point 0 would have been removed from \( B \) by a chord set \( K \in \mathcal{K}_{\sigma}(\partial B) \) with \( 0 \leq \sigma \leq 1 \) and \( 0 \in K \). We move the chord of \( K \) parallel until it contains 0 and construct this way a new chord set \( K' \in \mathcal{K}_{\sigma'}(\partial B) \) with even smaller area \( 0 \leq \sigma' \leq \sigma \). However, this contradicts \( B \in \mathcal{B}_{1}^{\text{aff}} \) which is characterized by \( \sigma' > 1 \) for any such chord set.

Finally, to show \( \mathcal{B} \sim \mathcal{B}_{1}^{\text{aff}} \), we consider \( B \in \mathcal{B} \), that is, \( 0 \in E_1(B) \). By (5) there is \( B' \in \mathcal{C}(\mathbb{R}^2) \) with \( 0 \in E_1(B') \subset B' \). Hence, by the definition (4) of the affine erosion for C-sets 0 cannot be contained in any chord set \( K \in \mathcal{K}_{\sigma}(\partial B') \) of area \( 0 \leq \sigma \leq 1 \). However, this means \( B' \in \mathcal{B}_{1}^{\text{aff}} \) and finishes the proof.

The dual morphological operator \( D_{\sigma} = E_{-\sigma} \), called the \( \sigma \)-affine dilation, is now given for u.s.c. functions \( u : \mathbb{R}^2 \to \mathbb{R}^2 \) as (see Moisan 1998, p. 417)

\[
D_{\sigma}(u) = -E_{\sigma}(-u) = \inf_{B \in \mathcal{B}^{\text{aff}}} \sup_{y \in B} u(x + y).
\]
Let us state the fundamental convergence result which we have taken from Cao (2003, Thm. 6.22); see also Moisan (1998, Thm. 4) or Guichard and Morel (1997, Thm. 10). A detailed account of the proof can be found in Cao (2003, §§6.4–5) or Guichard and Morel (2000, Chaps. 13–15 & 18).

**Theorem 2.** Let the propagator $T_h$ be one of the three operators

$$D_\sigma \circ E_\sigma, \quad E_\sigma \circ D_\sigma, \quad \text{or} \quad \frac{1}{2} \left( E_{2^{3/2}\sigma} + D_{2^{3/2}\sigma} \right); \quad h = \omega \sigma^{2/3}, \quad \omega = \frac{1}{2} \left( \frac{3}{2} \right)^{2/3}. $$

Define, for $u_0 \in \text{BUC}(\mathbb{R}^2)$, the time-discrete solution $u_h$ by the iteration

$$u_h(x, t) = T_h^n u_0(x), \quad t \in [nh, (n+1)h].$$

Then, as $h \to 0$, the discrete solution $u_h$ converges locally uniformly to the unique viscosity solution of the affine morphological scale space (AMSS).

A numerical algorithm implementing this approximation to the AMSS can be found in Moisan (1998, §IV) and Cao (2003, §6.6). Basically, it works on polygonal approximations of the level sets of $u$ and is therefore fairly expensive if applied to digital images: 8-bit gray-tone images have up to 256 different level sets that decompose into many different polygons to be dealt with.

### 4. Localization of Moisan’s Morphological Scheme.

The time propagator $T_h$ of Moisan’s (1998) scheme, which we have defined in Theorem 2, inherits from the affine erosion and dilation the affine invariance

$$T_h(u \circ \phi) = (T_h u) \circ \phi, \quad \phi \in \text{SL}(\mathbb{R}^2).$$

This is a direct consequence of the affine invariance of the structuring elements stated in Theorem 1,

$$\phi(\mathcal{B}^\text{aff}_\sigma) = \mathcal{B}^\text{aff}_\sigma, \quad \phi \in \text{SL}(\mathbb{R}^2).$$

Magnificent as this property is, there is a price to pay for it: since $\text{SL}(\mathbb{R}^2)$ contains arbitrarily distorting shears, $\mathcal{B}^\text{aff}_\sigma$ cannot be bounded. This causes problems, both in the technicalities of the theory and in the numerical realization of the scheme.

We will show in this section that a localization is possible if we sacrifice the affine invariance. Specifically, we consider the family of structuring elements

$$\mathcal{B}^\text{loc}_\sigma = \{ B \in \mathcal{B}^\text{aff}_\sigma : B \subset D(0, r(\sigma)) \}, \quad r(\sigma) = \kappa \sigma^\beta,$$

with appropriately chosen constants $\kappa > 0$ and $\beta \geq 0$. The catch is that the area of the disks $D(0, r(\sigma))$ will scale in $\sigma$ differently from the areas of the chord sets that build $\mathcal{B}^\text{aff}_\sigma$. Note that though the localized family is not affine invariant it is still isotropic,

$$\phi(\mathcal{B}^\text{loc}_\sigma) = \mathcal{B}^\text{loc}_\sigma, \quad \phi \in \text{SO}(\mathbb{R}^2).$$

We define the localized affine erosion and dilation by

$$E^\text{loc}_\sigma u(x) = \sup_{B \in \mathcal{B}^\text{loc}_\sigma} \inf_{y \in B} u(x+y), \quad D^\text{loc}_\sigma u(x) = \inf_{B \in \mathcal{B}^\text{loc}_\sigma} \sup_{y \in B} u(x+y).$$

We will generalize the convergence result of Theorem 2 to these localized operators. To this end we prove two lemmas that generalize the two key results for the affine
erosion and dilation in Cao's proof of the theorem, namely (2003, Prop. 6.18) and (2003, Lemma 6.20). Let us begin with the consistency of the two localized operators, compare Cao (2003, Prop. 6.18).

**Lemma 3.** Let $u$ be a $C^3$ function and $x$ a point of the plane such that $Du(x) \neq 0$ and $\text{curv} \ u(x) \neq 0$. If the exponent $\beta$ in the definition (8) of the localized structuring elements $B^\sigma$ is chosen according to $2/9 < \beta < 1/3$, then (with $\omega$ defined as in Theorem 2)

$$E^{\sigma}u(x) = u(x) + \omega \sigma^{2/3} |Du| \min(\text{curv}(u)^{1/3}, 0) + o(\sigma^{2/3}),$$

$$D^{\sigma}u(x) = u(x) + \omega \sigma^{2/3} |Du| \max(\text{curv}(u)^{1/3}, 0) + o(\sigma^{2/3}).$$

In both estimates the term $o(\sigma^{2/3})$ is locally uniform in $x$.

**Proof.** The proof of the corresponding result for $E^{\sigma}$ and $D^{\sigma}$ given by Cao (2003, Prop. 6.18, p. 120) shows that, after transformation to $x = 0$ and $u(0) = 0$, the arguments can be restricted to a disk $D(0, r)$ centered at $x = 0$ with a radius $r$ subject to the two conditions

$$\sigma^{1/3} = o(r), \quad r^3 = o(\sigma^{2/3}).$$

This localization is important to apply Taylor expansions. Based on the fact that the curvature of the level set $\chi_0$ can be bounded away from zero and chord sets can be controlled for disks, Cao then shows that all relevant chord sets will belong to $D(0, r)$ for sufficiently small $\sigma$. Hence, if we make $r = r(\sigma)$ satisfy the conditions (9), Cao's argument proves, in fact, that $E^{\sigma}u(x) = E_{\sigma}u(x)$ and $D^{\sigma}u(x) = D_{\sigma}u(x)$ for sufficiently small $\sigma$. Therefore, the localized operators inherit the consistency result from their nonlocalized counterpart.

Now, $r = r(\sigma)$ satisfies the first estimate in (9) if $0 \leq \beta < 1/3$ and the second one if $\beta > 2/9$. (Cao chooses $\beta = 1/4$, by the way.) $\square$

The second key result is about the control of the localized affine erosion and dilation for critical points. Because of the localization at hand it is, in fact, easier to prove than its nonlocalized counterpart considered by Cao (2003, Lemma 6.20).

**Lemma 4.** Let $u$ be a $C^2$ function and $x_0$ a critical point of the plane with $Du(x_0) = 0$ and $D^2u(x_0) = 0$. If the exponent $\beta$ in the definition (8) of the localized structuring elements $B^\sigma$ is chosen according to $0 \leq \beta \leq 1/3$ then

$$\lim_{x \to x_0, \sigma \to 0} \frac{E^{\sigma}u(x) - u(x)}{\sigma^{2/3}} = 0,$$

$$\lim_{x \to x_0, \sigma \to 0} \frac{D^{\sigma}u(x) - u(x)}{\sigma^{2/3}} = 0.$$

The limits are taken for $x$ and $\sigma$ independently.

**Proof.** We will follow an idea of Guichard and Morel who have proven a similar result (2000, Lemma 15.3) for affine invariant families of structuring elements. This will show that we have kept enough reminiscences of affine invariance to make the argument work.

The assertion at hand is easily established if we prove that there is a continuous function $G(Du, D^2u)$ such that $G(0, 0) = 0$ and

$$0 \leq D^{\sigma}u(x) - u(x) \leq G(Du(x), D^2u(x)) \sigma^{2/3} + o(\sigma^{2/3}),$$

where the term $o(\sigma^{2/3})$ is locally uniform in $x$; by duality we then have

$$0 \geq E^{\sigma}u(x) - u(x) \geq -G(-Du(x), -D^2u(x)) \sigma^{2/3} + o(\sigma^{2/3}).$$
To prove (10) we choose a point \( x \) and assume, without loss of generality by the isotropy of \( B^\text{loc} \), that \( Du(x) = (p, 0) \). Taylor expansion thus gives, locally uniform,

\[
    u(x + y) - u(x) = p y_1 + a y_1^2 + b y_1 y_2 + c y_2^2 + o(|y|^2).
\]

Each chord set of the disk \( D(0, r(\sigma)) \) with 0 on its chord must be a semi-disk of area \( \pi r(\sigma)^2 / 2 = \pi \kappa^2 \sigma^2 / 2 \). Thus, since \( 2\beta \leq 2/3 < 1 \), such a chord set has an area larger than \( \sigma \) for \( \sigma \) sufficiently small, \( 0 < \sigma \leq \sigma_* \) say. That is, \( B_* = D(0, r(\sigma_*)) \in B^\text{loc} \) in particular. By \( \beta \leq 1/3 \) we get \((\sigma / \sigma_*)^1/3 B_* \subset D(0, r(\sigma)) \) for \( 0 < \sigma \leq \sigma_* \). Thus, because of the affine invariance of the unconstrained structuring elements stated in Theorem 1, we have that

\[
    B_\sigma = \left( (\sigma / \sigma_*)^{2/3} \right) B_* \in B^\text{loc}.
\]

Note that there is a certain amount of shear involved in the definition of \( B_\sigma \). For \( y = (y_1, y_2) \in B_\sigma \) we get the bounds \( |y_1| \leq \mu^2 \sigma^{2/3} \) and \( |y_2| \leq \mu \sigma^{1/3} \) with a constant \( \mu \) depending on \( \kappa \), \( \beta \), and \( \sigma_* \). Using the Taylor expansion (11) we obtain

\[
    0 \leq D^\text{loc}_\sigma u(x) - u(x) = \inf_{B \in B^\text{loc}} \sup_{y \in B} (u(x + y) - u(x)) \leq \sup_{y \in B_\sigma} (p y_1 + a y_1^2 + b y_1 y_2 + c y_2^2) + o(\sigma^{2/3}),
\]

\[
    \leq (|p| + |c|) \mu^2 \sigma^{2/3} + o(\sigma^{2/3}) \leq \mu^2 (\| D u(x) \| + \| D^2 u(x) \|) \sigma^{2/3} + o(\sigma^{2/3}),
\]

with appropriately chosen norms of the gradient and Hessian of \( u \). Summarizing, we can take \( G(Du, D^2 u) = \mu^2 (\| D u \| + \| D^2 u \|) \), which has the asserted properties. \( \square \)

These two lemmas in hand we can proceed word-for-word as in Cao’s (2003, Thm. 6.22) proof of Theorem 2 to establish the following convergence result.\(^2\)

**Theorem 5.** Let be \( \kappa > 0 \) and \( 2/9 < \beta < 1/3 \) in the definition (8) of the localized structuring elements \( B^\text{loc} \). Let the propagator \( T_h \) be one of the three operators

\[
    D^\text{loc}_\sigma \circ E^\text{loc}_\sigma, \quad E^\text{loc}_\sigma \circ D^\text{loc}_\sigma, \quad \text{or} \quad \frac{1}{2} \left( E^\text{loc}_{2/3/\sigma} + D^\text{loc}_{2/3/\sigma} \right); \quad h = \omega \sigma^{2/3}, \quad \omega = \frac{1}{2} \left( \frac{3}{2} \right)^{2/3}.
\]

Define, for \( u_0 \in \text{BUC}(\mathbb{R}^2) \), the time-discrete solution \( u_h \) by the iteration

\[
    u_h(x, t) = T_h^n u_0(x), \quad t \in [nh, (n + 1)h].
\]

Then, as \( h \to 0 \), the discrete solution \( u_h \) converges locally uniformly to the unique viscosity solution of the affine morphological scale space (AMSS).

\(^2\)Alternatively, a proof can be based on a convergence result of the Barles–Souganidis (1991) type as stated and proved in Guichard and Morel (2000, Prop. 17.16). This is, because \( D_\sigma \), \( E_\sigma \), and therefore \( T_h \) satisfy a uniform local comparison principle (Guichard and Morel 2000, Def. 17.14) which is easily established since, for \( \sigma \leq \sigma_* \), the structuring elements of the families \( B^\text{loc} \) are contained in the disk \( D(0, r(\sigma_*)) \). Now, our Lemmas 3 & 4 establish the uniform consistency in the sense of Guichard and Morel (2000, Def. 17.13) except for the case \( Du(x) \neq 0 \) and \( \text{curv} u(x) = 0 \) that, however, can be dealt with on the side of the test function \( f \) in Guichard and Morel’s (2000, p. 208) proof by replacing it by \( f_\epsilon(y) = f_\epsilon(x) + \epsilon |y - x|^2 \). (Letting \( \epsilon \to 0 \) only after taking the limit \( \sigma \to 0 \); see the argument given by Cao (2003) in the paragraph following Eq. (6.28).)
5. A Family of Local Morphological Image Operators. We will now study a different type of localization of the families $B_{\sigma}^{aff}$, namely the one obtained by discretizing them on the $3 \times 3$ block of pixels that is frequently used in digital image processing. As we will see, already for this smallest case there is a rich structure surfacing. Certainly, convergence of the resulting schemes will be no issue here. However, being defined as coarse discretizations the obtained image operators are closely related to the AMSS and may inherit some of its favorable properties. It is further of interest to compare the new image operators with known ones that are related to the mean curvature motion.

Pixels and Digital Images. We call the half-open square $\left[ -\frac{1}{2}, \frac{1}{2} \right]^2$ and its integer translates centered at the grid $\mathbb{Z}^2$ a pixel. We choose 8-connectivity of pixels, see Soille (2003, §2.6.2). An 8-bit gray-tone digital image, extended by reflection and symmetry from its original rectangular domain of definition, can be considered as a function $u : \mathbb{Z}^2 \rightarrow \{0, \ldots, 255\}$. By further extending it as a constant from the grid points to the surrounding pixel we obtain a piecewise constant function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$.

We define $P_{3 \times 3}$ as the collection of the 512 different sets (binary images) made by joining some of the pixels that belong to the $3 \times 3$ neighborhood of the pixel centered at 0. The subcollection of the 256 sets that contain the center pixel at 0 is denoted by $P_{3 \times 3}^\circ$; see Soille (2003, Fig. 5.10) for a complete listing.

The Median Filter and Image Operators Based on Mean Curvature Motion. The best known nonlinear filters in image processing are the median filters. According to Gonzalez and Woods (2002, p. 123) they are quite popular because, for certain types of random noise, they provide excellent noise-reduction capabilities, with considerable less blurring than linear smoothing filters of similar size. Median filters are particularly effective in the presence of impulse noise, also called salt-\&-pepper noise because of its appearance as white and black dots superimposed on an image.

The self-dual $3 \times 3$-median filter (Soille 2003, §3.5.4) can straightforwardly be written in the form (see Guichard and Morel 2000, Rem. 10.11)\(^4\)

$$M_{3 \times 3}^3(x) = \sup_{B \in \mathcal{B}_{3 \times 3}^\circ} \inf_{y \in B} u(x + y) = \inf_{B \in \mathcal{B}_{3 \times 3}^\circ} \sup_{y \in B} u(x + y),$$

where

$$\mathcal{B}_{3 \times 3}^\circ = \{ B \in P_{3 \times 3} : |B| = 5 \} \times \mathcal{B}_{3 \times 3}^\circ = \{ B \in P_{3 \times 3} : |B| \geq 5 \}.$$

Note that $\# \mathcal{B}_{3 \times 3}^\circ = \binom{9}{5} = 126$. As pointed out by several authors,\(^5\) the median filter can be viewed as the time propagator of a discretization of the mean curvature motion (MCM).

On the other hand, Catté et al. (1995) started with considering the mean curvature motion and have obtained a convergence theorem similar to Theorem 2 based on the the structuring elements

$$B_{Catté}^\sigma = \{ B : B \text{ is a line segment of length } \sigma \text{ centered at } 0 \}.$$ 

Note that the family $B_{Catté}^\sigma$ resolves all possible angular directions, a feature, that is common to all monotone numerical schemes of the mean curvature motion known to

---

\(^3\)In fact, the study of convergence corresponds to the analysis of $k \times k$-blocks of pixels as $k \rightarrow \infty$.

\(^4\)Like $u$ the resulting functions are digital images, that is, they are constant on each pixel.

\(^5\)Guichard and Morel (1997, §1.4) and (2000, Chap. 10), or Oberman (2004, §1.5), to name just a few.
be convergent, see, e.g., Crandall and Lions (1996) or Oberman (2004).\footnote{In particular, Crandall and Lions (1996, §3.1) argue that this is indeed necessary because of the possible degeneracy of the nonlinear elliptic operator $|Du| \operatorname{curv} u$. In fact, Motzkin and Wasow (1953) proved already for the linear case that better and better angular resolutions, namely larger and larger finite-difference stencils, are needed for larger and larger condition numbers of the elliptic operator at hand.}

Finally, Catté et al. (1995, §7) have suggested a coarse discretization $\mathcal{B}_{\text{Catté}}^{3 \times 3}$ of $\mathcal{B}_{\text{Catté}}$ by just taking the 4 sets $B \in \mathbb{P}_{\text{sym}}^{3 \times 3}$, of three pixels each, that are aligned to one of the symmetry axes of the $3 \times 3$ block; see Fig. 2.i. The corresponding image operators are obtained by alternating, or averaging, the two operators

$$
E_{\text{Catté}}^{3 \times 3} u(x) = \sup_{B \in \mathcal{B}_{\text{Catté}}^{3 \times 3}} \inf_{y \in B} u(x + y), \quad D_{\text{Catté}}^{3 \times 3} u(x) = \inf_{B \in \mathcal{B}_{\text{Catté}}^{3 \times 3}} \sup_{y \in B} u(x + y).
$$

**A Family of Image Operators Based on the AMSS.** To obtain image operators from the AMSS we follow the route taken by Catté et al. (1995) on their way from the continuous $\mathcal{B}_{\text{Catté}}$ to the discrete $\mathcal{B}_{\text{Catté}}^{3 \times 3}$: first discretize the slopes that define the structuring elements before restricting them to blocks of pixels. We begin with making the slopes of the chords explicit that define $\mathcal{B}_{\text{aff}}^{3 \times 3}$:

$$
\mathcal{B}_{\text{aff}}^{3 \times 3} = \bigcap_{0 \leq \theta < \pi} \mathcal{B}_{\theta, \text{aff}}^{3 \times 3}
$$

where $\mathcal{B}_{\theta, \text{aff}}^{3 \times 3}$ is defined like $\mathcal{B}_{\theta, \text{aff}}$ but for the difference that only those chord sets are considered whose chord makes an angle $\theta$ with the horizontal line at 0. Now, for the angle $\theta \in \Theta_{\text{sym}}$ of one of the 4 symmetry axes of the $3 \times 3$ block, we would like to restrict the structuring elements of $\mathcal{B}_{\theta, \text{aff}}^{3 \times 3}$ by selecting those that belong to $\mathbb{P}_{\text{sym}}^{3 \times 3}$. However, there is the minor technical problem that none of the sets $B \in \mathbb{P}_{\text{sym}}^{3 \times 3}$ is a C-set. This changes immediately to the positive if we take the dilation $B_{\epsilon}$ of $B$ by an open disk of radius $\epsilon$ instead. Note that $B = \bigcap_{\epsilon > 0} B_{\epsilon}$ for $B \in \mathbb{P}_{\text{sym}}^{3 \times 3}$. Hence, we define

$$
\mathcal{B}_{\theta, \text{aff}}^{3 \times 3} = \{ B \in \mathbb{P}_{\text{sym}}^{3 \times 3} : B_{\epsilon} \in \bigcap_{0 \leq \theta < \pi} \mathcal{B}_{\theta, \text{aff}}^{3 \times 3} \text{ for all } \epsilon > 0 \}.
$$

One easily sees that

$$
\mathcal{B}_{\theta, \text{aff}}^{3 \times 3} = \{ B \in \mathbb{P}_{\text{sym}}^{3 \times 3} : |B \cap H| \geq \sigma \text{ for each half plane } H \in \mathbb{H}_{\text{sym}} \},
$$

where $\mathbb{H}_{\text{sym}}$ is the set of the 8 half-planes that are bounded by one of the 4 symmetry axes of the $3 \times 3$ block. Using a short computer program\footnote{Which can be found, together with all the code and images discussed further below, at the URL http://www-m3.ma.tum.de/~bornemann/AMSSCodeAndData.zip.} that helps us with...
checking just finitely many cases we can show that the admissible range \([0, \frac{9}{2}]\) of the area parameter \(\sigma\) is divided into 8 parts, each of which gives a different set \(\mathcal{B}_{\sigma}^{3\times3}\) of structuring elements, see Table 1. If convenient we denote these families, and the image operators generated by them, by replacing the index \(\sigma\) with the roman numeral shown in this table. Note that the basis families \(\mathcal{B}_{\sigma}^{3\times3}\) contain considerably fewer elements than \(\mathcal{B}_{\sigma}^{3\times3}\); all basis elements are listed in Fig. 2.a–h. The corresponding image operators are obtained by alternating, or averaging, the \(3\times3\)-affine erosion \(E_{\sigma}^{3\times3}\) and dilation \(D_{\sigma}^{3\times3}\) defined by

\[
E_{\sigma}^{3\times3} u(x) = \sup_{B \in \mathcal{B}_{\sigma}^{3\times3}} \inf_{y \in B} u(x + y), \quad D_{\sigma}^{3\times3} u(x) = \inf_{B \in \mathcal{B}_{\sigma}^{3\times3}} \sup_{y \in B} u(x + y).
\]

**Mutual Relations between the Families of 3 \times 3-Structuring Elements.** A closer look at Fig. 2 teaches that there are interesting order relations between the discrete families of structuring elements considered so far, namely

\[
\mathcal{B}_{\text{VIII}}^{3\times3} \preceq \mathcal{B}_{\text{VII}}^{3\times3} \preceq \mathcal{B}_{\text{VI}}^{3\times3} \preceq \mathcal{B}_{\text{V}}^{3\times3} \preceq \mathcal{B}_{\text{IV}}^{3\times3} \preceq \mathcal{B}_{\text{med}}^{3\times3}; \quad (12)
\]

and

\[
\mathcal{B}_{\text{V}}^{3\times3} \preceq \left\{ \mathcal{B}_{\text{IV}}^{3\times3}, \mathcal{B}_{\text{Catté}}^{3\times3} \right\} \preceq \mathcal{B}_{\text{III}}^{3\times3} \preceq \mathcal{B}_{\text{II}}^{3\times3} \preceq \mathcal{B}_{\text{I}}^{3\times3}. \quad (13)
\]
Now, the single element of $\mathbb{B}_{3 \times 3}^I$ is just the pixel centered at 0. Therefore, we obviously have

$$E_{3 \times 3}^I u(x) = D_{3 \times 3}^I u(x) = u(x).$$

On the other hand, the single element of $\mathbb{B}_{3 \times 3}^{VIII}$ is the $3 \times 3$ square. Denoting by $E_{3 \times 3}$ and $D_{3 \times 3}$ the standard morphological erosion and dilation by the $3 \times 3$ square (see Soille 2003, Chap. 3), we thus have

$$E_{3 \times 3}^{VIII} = E_{3 \times 3}, \quad D_{3 \times 3}^{VIII} = D_{3 \times 3}.$$

In particular, $D_{3 \times 3}^{VIII} \circ E_{3 \times 3}^{VIII}$ is the morphological opening and $E_{3 \times 3}^{VIII} \circ D_{3 \times 3}^{VIII}$ the morphological closing by the $3 \times 3$ square, see Soille (2003, Chap. 4). Both are idempotent operators, that is, iteration does not change the result.

Remarks on the Efficient Implementation. For a given structuring element $B \in \mathbb{B} \subset \mathbb{P}^{3 \times 3}$ the operators

$$E_B u(x) = \inf_{y \in B} u(x + y), \quad D_B u(x) = \sup_{y \in B} u(x + y),$$

are just the erosion and dilation of grey-tone images by the structuring element $B$ as they are commonly used in mathematical morphology, see Soille (2003, Chap. 3). For these operators, and more general any rank filter that assign the kth value in the order of the values $u(x + y)$, $y \in B$, between minimum and maximum, fast implementations based on the moving histogram technique are available. In particular, this technique also applies to the rank filter that takes the medial value, that is, the median filter. Details on the moving histogram technique can be found in Guichard and Morel (1997, §4) or Soille (2003, §3.9.3). Matlab’s image processing toolbox and the

```matlab
function img = ED(img,B,n)
    ero = @imerode; dil = @imdilate; % SDC Morphology Toolbox
    ero = @mmero; dil = @mmdil; % Image Processing Toolbox
    function out = E(in)
        out = ero(in,B(:,:,1));
        for k=2:size(B,3)
            out = max(out,ero(in,B(:,:,k)));
        end
    end
    function out = D(in)
        out = dil(in,B(:,:,1));
        for k=2:size(B,3)
            out = min(out,dil(in,B(:,:,k)));
        end
    end
    for k=1:n
        img = E(D(img));
    end
end
```

Fig. 3. A Matlab implementation of $(E_B \circ D_B)^n$ for a family $B$ of structuring elements stored in the array $B$. 
yields the straightforward implementation of, say, the iterated operator (shown in Fig. 3. Here, the family $B$ that, the representation $8$ SDC Morphology Toolbox $E$ where “pepper” (the black dots) is removed. Exactly this behavior can be observed in Fig. 4 the large white mushy region generated by the morphological opening while the dots) that still remains with the iterated median filter tends to grow until it becomes 

us to expect that, for these families, the iteration of $E$ can be stored once and for all in a look-up table. Then, the calculation of $E$ and $D$ just requires a well-organized table look-up.

Table 2 shows the CPU times of the various implementations for 10 iterations of the image operator $E_B \circ D_B$ applied to an image of size $1024 \times 1024$ using various implementations. 

<table>
<thead>
<tr>
<th>Implementation</th>
<th>$E_B^3 \circ D_B^3$ Catté</th>
<th>$E_B^3 \circ D_B^3$ III</th>
<th>$E_B^3 \circ D_B^3$ IV</th>
<th>$M^3 \circ M^3$</th>
<th>$M^5 \circ M^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ED using Image Processing Toolbox</td>
<td>12 sec</td>
<td>40 sec</td>
<td>120 sec</td>
<td>255 sec</td>
<td>—</td>
</tr>
<tr>
<td>ED using Morphology Toolbox</td>
<td>2.0 sec</td>
<td>6.9 sec</td>
<td>20 sec</td>
<td>42 sec</td>
<td>—</td>
</tr>
<tr>
<td>Table look-up</td>
<td>8.4 sec</td>
<td>8.4 sec</td>
<td>8.4 sec</td>
<td>4.2 sec</td>
<td>—</td>
</tr>
<tr>
<td>Matlab’s medfilt2</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>1.4 sec</td>
<td>3.4 sec</td>
</tr>
</tbody>
</table>

SDC Morphology Toolbox $^8$ for Matlab offer implementations of $E_B$ and $D_B$. Given that, the representation 

$$E_B u(x) = \sup_{B \in B} E_B u(x), \quad D_B u(x) = \inf_{B \in B} D_B u(x),$$

yields the straightforward implementation of, say, the iterated operator $(E_B \circ D_B)^n$ shown in Fig. 3. Here, the family $B$ of structuring elements is stored as a $3 \times 3 \times \#B$ logical array $B$ in Matlab: $B(i,j,k)$ is true if the pixel $ij$ is contained in the $k$th structuring element, and false otherwise. The center pixel corresponds to $i = j = 2$.

However, for a large number $\#B$ of structuring elements there is a more effi- cient way of calculating $E_B$ and $D_B$. To this end we note that the results $E_B u(x)$ and $D_B u(x)$ depend on the particular order pattern of the values in the $3 \times 3$-neighborhood of $x$ only. There are exactly $9! = 362880$ such patterns which, together with the position of the result, can be stored once and for all in a look-up table. Then, the calculation of $E_B$ and $D_B$ just requires a well-organized table look-up.

Table 2 shows the CPU times of the various implementations for 10 iterations of the image operator $E_B \circ D_B$ applied to an image of size $1024 \times 1024$. We see that the implementation of the standard erosions and dilations in the SDC Morphology Toolbox is, by a factor of about 6, more efficient than the one of Matlab’s own image processing toolbox. On the other hand, our table look-up outperforms the straightforward implementation for $\#B > 14$. Of course, the moving histogram technique used in Matlab’s medfilt2 implementation of the median filter is much faster than our table look-up.

An Illustrative Application: Denoising Salt-&-Pepper Noise. We study the application of the various image operators, which we have introduced so far, to the denoising of an image affected by 60% salt-&-pepper noise, shown in Fig. 5.a. In particular, we raise the question whether any of the new $3 \times 3$ image operators gives results better than the iterated $3 \times 3$-median filter.

The ordering (12) of the families $B^3_{\text{VIII}}, \ldots, B^3_{\text{IV}}$ between $B^3_{\text{III}}$ and $B^3_{\text{med}}$ leads us to expect that, for these families, the iteration of $E_B \circ D_B$ performs qualitatively in between the results of the iterated median filter $M^3 \circ M^3$ and the closing $E^3_{\text{VIII}} \circ D^3_{\text{VIII}} = E_B \circ D_B$ by the $3 \times 3$ square. More precisely we expect that the “salt” (the white dots) that still remains with the iterated median filter tends to grow until it becomes the large white mushy region generated by the morphological opening while the “pepper” (the black dots) is removed. Exactly this behavior can be observed in Fig. 4 where $E^3_{\text{IV}} \circ D^3_{\text{IV}}$ and $M^3 \circ M^3$ have been iterated until the results become (visually)


$^9$All experiments were performed on a notebook with a 2GHz Pentium M CPU and 1 GB of RAM.
stationary. Dually, $D_B \circ E_B$ performs between the median filter and the opening by the $3 \times 3$ square, that is, “pepper” is enhanced and “salt” removed. Summarizing, none of these image operators gives results better than the iterated median filter.

On the other hand the ordering (13) of the families $\hat{B}_3^{3 \times 3}$ and $\hat{B}_3^{3 \times 3}$ between $\hat{B}_3^{3 \times 3}$ and $\hat{B}_3^{3 \times 3}$ Catté and their mutual incomparability with $\hat{B}_3^{3 \times 3}$ med shows that there are three interesting families to compare with the median filter: $\hat{B}_3^{3 \times 3}$, $\hat{B}_3^{3 \times 3}$, and $\hat{B}_3^{3 \times 3}$ Catté. (Recall that $E_3^{3 \times 3}$ and $D_3^{3 \times 3}$ both give the identity operator.) We compare the results of iterating $E_B \circ D_B$ and $D_B \circ E_B$ until they become (visually) stationary, in Fig. 5 applied to the noisy image and, in Fig. 6, applied to the original image unaffected by noise. We have included in this comparison the averaged image operators $\frac{1}{2}(E_B + D_B)$ iterated, to become comparable, to twice as many iterations as used for $E_B \circ D_B$ and $D_B \circ E_B$. The result of this comparison is that among the $3 \times 3$ image operators the families $\hat{B}_3^{3 \times 3}$ Catté and $\hat{B}_3^{3 \times 3}$ certainly give the best denoising results, with an advantage to our new family $\hat{B}_3^{3 \times 3}$, which is related to the AMSS instead of the mean curvature motion as $\hat{B}_3^{3 \times 3}$ Catté. Though the remaining family $\hat{B}_3^{3 \times 3}$ is much less diffusive (compare Fig. 5.d,g, & j, or Fig. 6.d,g, & j) its noise-reduction capabilities are slightly inferior to the iterated median filter.

Further, if we compare with the iterated $5 \times 5$-median filter, the iteration of $D_3^{3 \times 3} \circ D_3^{3 \times 3}$ or $D_3^{3 \times 3} \circ E_3^{3 \times 3}$ gives nearly comparable results for the denoising while being less diffusive otherwise (compare, e.g., the eyelashes in the processing of the original image shown in Fig. 6.c & f). Thus, it has to be expected that an extension of our study to the next larger case of the $5 \times 5$-block of pixels will uncover further interesting morphological image operators. However, there will be a considerable increase of the combinatorial complexity involved and, more important, the question of a really efficient implementation, so that these image operators can compete with the median filter in practice, becomes a major issue. We leave this to future work.

References.


Fig. 5. Denoising: comparison of median filters with the image operators generated by $B_{3 \times 3}^{\text{Catté}} \triangleright B_{3 \times 3}^{\text{III}} \triangleright B_{3 \times 3}^{\text{II}}$. 
Fig. 6. The same image operators as in Fig. 5, applied to the original image unaffected by noise.