Solution of Trefethen's Problem 1

J. Boersma and J.K.M. Jansen
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Trefethen's Problem 1 as proposed in [1] reads:
What is $\lim_{\varepsilon\to 0}\int_\varepsilon^1 x^{-1}\cos(x^{-1}\log x)\,dx$?

Solution. Replace x by $1/x$, then the question is to evaluate $\int_1^\infty x^{-1}\cos(x\log x)\,dx$. By contour integration in the complex z-plane one has

$$\int_1^\infty x^{-1}e^{ix\log x}\,dx = \int_C z^{-1}e^{iz\log z}\,dz + \int_i^{i\infty} z^{-1}e^{iz\log z}\,dz,$$

where C is the arc of the circle $|z| = 1$ from $z = 1$ to $z = i$. In the two integrals on the right, parametrize $z = e^{it}$, $0 \le t \le \pi/2$, and $z = iy$, $y \ge 1$. By taking real parts it follows that

$$\int_1^\infty x^{-1}\cos(x\log x)\,dx = \int_0^{\pi/2} e^{-t\cos t}\sin(t\sin t)\,dt + \int_1^\infty y^{-1}e^{-y\log y}\cos(\pi y/2)\,dy.$$

Both integrals on the right have been numerically evaluated by means of Mathematica, leading to

$$\int_1^\infty x^{-1}\cos(x\log x)\,dx = 0.32336743167777876$$

with 17 correct digits.

Reference

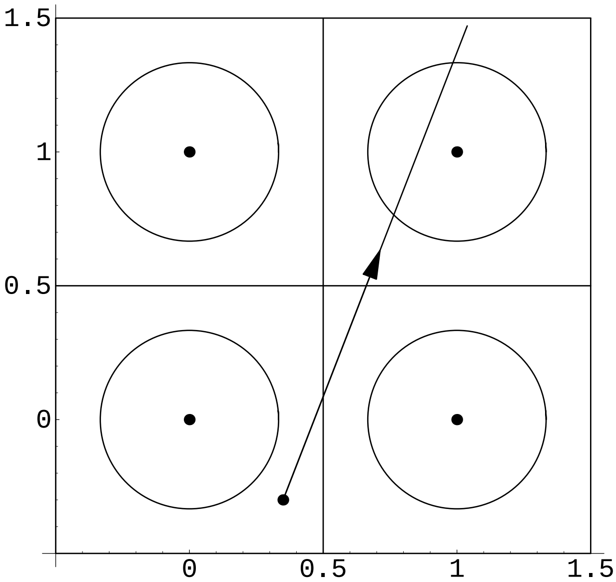
[1] L.N. Trefethen, The hundred-dollar, hundred-digit challenge problems, SIAM News, Vol. 35, Number 1, p. 3, January/February 2002.

■ Solution of Trefethen's Problem 2 by F.H. Simons

The problem is taken from L.N. Trefethen's list in SIAM News **35**, 1 (2002), problem 2.

A photon moving at speed 1 in the x-y plane starts at $t = 0$ at $(x, y) = (0.5, 0.1)$ heading due east. Around every integer lattice point (i, j) in the plane, a circular mirror of radius $1/3$ has been erected. How far from the origin is the photon at $t = 10$?

```
<< Graphics`Arrow`
p = {0.35, -0.3};
v = {Cos[1.2], Sin[1.2]};
gr = { Table[Line[ {{-1/2, i}, {3/2, i}}], {i, -1/2, 3/2}],
   Table[Line[{{i, -1/2}, {i, 3/2}}], {i, -1/2, 3/2}],
    Table[ Circle[{i, j}, 1/3], {i, 0, 1}, {j, 0, 1}],
   {PointSize[0.02], Point[p], Table[Point[{i, j}], {i, 0, 1}, {j, 0, 1}]},
   Arrow[p, p+v],
   Line[{p, p+1.9 v}]};
Show[Graphics[gr], AspectRatio → Automatic,
 Axes → True, AxesOrigin → {-1/2, -1/2}]
```




```
- Graphics -
```

Roughly speaking, we solve the problem in the following way. Consider the half line starting at p in the direction of v. That half line consists of intervals, cut off by the unit squares with center a lattice point, as indicated in the figure. For each point on the half line we find the corresponding center by rounding the position to the nearest integers. It is easily verified that the half line does not intersect the corresponding circle when the length of such an interval is less than 2/3 (actually $\sqrt{2} - 2/3$) . Hence

When the half line starting at p in the direction of v does not intersect the circle corresponding to p, then the first circle that might be intersected is the circle corresponding to $p + 2\,v/3$.

When the circle is intersected, we determine the first point of intersection and the new direction. Otherwise we simply shift to $p + 2\,v/3$.

Finding the new direction is done by a matrix multiplication; (a, b) is the axis.

```
mat[{a_, b_}] = Drop[#, 2] & /@ RowReduce[ {{a, b, -a, -b}, {-b, a, -b, a}}]
```

$$\left\{\left\{\frac{-a^2+b^2}{a^2+b^2}, -\frac{2\,a\,b}{a^2+b^2}\right\}, \left\{-\frac{2\,a\,b}{a^2+b^2}, \frac{a^2-b^2}{a^2+b^2}\right\}\right\}$$

We compute the distance of the origin to the position of the point at $t = 10$, using arbitrary precision numbers, in the following way.

```
p = N[{1 / 2, 1 / 10}, 70];
v = {1, 0};
resttime = 10;
While[resttime > 0,
  m = Round[p + 2 v / 3];
  t0 = Min[ Cases[t /. Solve[(p + t v - m ). (p + t v - m ) == 1 / 9, t] , _?Positive]];
  If[t0 < resttime,
   p = p + t0 v; v = mat[p - m].v,
   t0 = Min[resttime, 2 / 3];
   p = p + t0 v];
  resttime = resttime - t0];
Sqrt[p.p]
```

```
0.99526291944335416089031180942672
```

Solution of Trefethen's Problem 3

J.K.M. Jansen and J. Boersma
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Trefethen's Problem 3 as proposed in [1] reads:
The infinite matrix A with entries $a_{11} = 1$, $a_{12} = 1/2$, $a_{21} = 1/3$, $a_{13} = 1/4$, $a_{22} = 1/5$, $a_{31} = 1/6$, etc., is a bounded operator on ℓ^2. What is $\|A\|$?

Solution. The elements of the matrix A are generally given by

$$a_{mn} = 2/[(m+n)^2 - m - 3n + 2], \quad m, n = 1, 2, 3, \ldots.$$

The norm $\|A\|$ is equal to the largest singular value σ, say, of A. Let $A^{(N)}$ denote the $N \times N$ matrix obtained by truncation of A. By a singular value decomposition we determine the largest singular value $\sigma^{(N)}$ of $A^{(N)}$, for $N = 100, 200, 400, 800, 1600$. The differences in the table of $\sigma^{(N)}$ suggest that $\sigma - \sigma^{(N)} = O(1/N^3)$. By an appropriate extrapolation we then find

$$\|A\| = \sigma = 1.2742241528$$

to 10 decimal places.

Reference

[1] L.N. Trefethen, The hundred-dollar, hundred-digit challenge problems, SIAM News, Vol. 35, Number 1, p. 3, January/February 2002.

Solution of Trefethen's Problem 4

J. Boersma and J.K.M. Jansen
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Trefethen's Problem 4 as proposed in [1] reads:
What is the global minimum of the function
$\exp(\sin(50x)) + \sin(60e^y) + \sin(70\sin(x)) + \sin(\sin(80y)) - \sin(10(x+y))$
$+ \frac{1}{4}(x^2 + y^2)$?

Solution. Denote the function by $f(x,y)$. The global minimum of $f(x,y)$ is determined by means of Mathematica. First, by means of the procedure `DensityPlot` it is found that $f(x,y) \leq -3$ occurs on the squares $S_1 = \{(x,y) | -0.4 \leq x \leq -0.38,\ -0.1 \leq y \leq -0.08\}$ and $S_2 = \{(x,y) | -0.04 \leq x \leq -0.02,\ 0.2 \leq y \leq 0.22\}$ only. A 3D-Plot shows that $f(x,y)$ has a single minimum on S_1 and on S_2. Finally, by the procedure `FindMinimum` it is found that the global minimum is taken on S_2 and its value is

$$-3.306868647475237,$$

to 15 decimal places.

Reference

[1] L.N. Trefethen, The hundred-dollar, hundred-digit challenge problems, SIAM News, Vol. 35, Number 1, p. 3, January/February 2002.

Solution of Trefethen's Problem 5

J.K.M. Jansen and J. Boersma
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Trefethen's Problem 5 as proposed in [1] reads:
Let $f(z) = 1/\Gamma(z)$, where $\Gamma(z)$ is the gamma function, and let $p(z)$ be the cubic polynomial that best approximates $f(z)$ on the unit disk in the supremum norm $\|\cdot\|_\infty$. What is $\|f-p\|_\infty$?

Solution. By means of the maximum modulus principle it follows that $\max_{|z|\le 1}|f(z)-p(z)|$ is attained on the boundary $|z|=1$. Denote the best cubic polynomial approximation to $f(z)$, by $p^*(z)$. Then according to a theorem of Tonelli [2, pp. 446-448] one has: (i) $p^*(z)$ is unique; (ii) $\max_{|z|=1}|f(z)-p^*(z)|$ is attained at at least 5 points of the unit circle $|z|=1$. Item (i) implies that $p^*(z)$ has real coefficients.
Introduce the notations

$$p(z) = a + b\,z + c\,z^2 + d\,z^3, \quad w(z) = f(z) - p(z),$$

$$M = M(a,b,c,d) = \max_{|z|=1}|w(z)|.$$

Our colleague F.H. Simons developed a Mathematica procedure to determine the (global) maximum $M(a,b,c,d)$ for real a,b,c,d. Next we searched for the minimum of $M(a,b,c,d)$ by use of the procedure `FindMinimum`. By a trial-and-error approach we found a polynomial $p(z)$ for which $\max_{|z|=1}|w(z)| = M = 0.2148099883$ is attained at 5 points $z = z_k$, $k = 1,2,\ldots,5$, with $|z_k| = 1$, $\operatorname{Im} z_1 > 0$, $\operatorname{Im} z_2 > 0$, $z_3 = -1$, $z_4 = \overline{z_2}$, $z_5 = \overline{z_1}$. Now the present polynomial is *not* the best polynomial. This follows from a lemma of Vidensky [2, pp. 450-452] which in the present case takes the form: For the polynomial $p(z)$ to be the polynomial of best approximation for $f(z)$ on $|z|=1$, it is necessary and sufficient that (i) $|w(z_k)| = M$ for $k = 1,2,\ldots,5$; (ii) $\arg[(-1)^{k+1}\overline{w}(z_k)/\Delta_k] = 0$ for $k = 1,2,\ldots,5$, where Δ_k is the 4×4 Vandermonde determinant with rows $[1 \;\; z_j \;\; z_j^2 \;\; z_j^3]$, $j = 1,2,\ldots,5$, $j \neq k$. (Notice that we corrected a printing error: $w(z_k)$ in the original formulation of condition (ii) has been corrected into $\overline{w}(z_k)$, as stated in the proof of the

lemma.) Clearly, the previous polynomial $p(z)$ satisfies condition (i), but condition (ii) turns out to be violated.
To determine the best polynomial we employed the following algorithm based on the necessary and sufficient conditions mentioned before. Let $z_k = e^{2\pi i t_k}$, $k = 1, 2, \ldots, 5$, with $0 < t_1 < 1/2$, $0 < t_2 < 1/2$, $t_3 = 1/2$, $t_4 = 1 - t_2$, $t_5 = 1 - t_1$. Introduce the system of 7 (nonlinear) equations

$$|w(z_k)| = M, \text{ for } k = 1, 2; \quad w(-1) = -a + b - c + d = -M;$$

$$\frac{d}{dt}|w(e^{2\pi it})|^2 = -4\pi \operatorname{Im}[e^{2\pi it}\, w(e^{-2\pi it})\, w'(e^{2\pi it})] = 0, \text{ for } t = t_1,\ t = t_2;$$

$$\arg[(-1)^{k+1}\overline{w}(z_k)/\Delta_k] = 0 \text{ for } k = 1, 2;$$

for the 7 unknowns M, a, b, c, d, t_1, t_2. This system is rapidly solved (in 17 seconds with 30 digits' accuracy) by the procedure `FindRoot`. In this manner we found the cubic polynomial

$$\begin{aligned} p(z) &= 0.005541950733 + 1.019761852984\, z \\ &\quad +0.625211916433\, z^2 - 0.603343220408\, z^3 \end{aligned}$$

for which $\max_{|z|=1} |f(z) - p(z)| = 0.21433523459$ is attained at 5 points on $|z| = 1$; also condition (ii) turns out to be satisfied. We conclude that $p(z)$ is sufficiently close to the best polynomial $p^*(z)$ to make sure that

$$\|f - p^*\|_\infty = 0.21433523459$$

to 11 decimal places. Trefethen [3, p. 348] reports the less accurate value $\|f - p^*\|_\infty = 0.217$.

References

[1] L.N. Trefethen, The hundred-dollar, hundred-digit challenge problems, SIAM News, Vol. 35, Number 1, p. 3, January/February 2002.
[2] V.I. Smirnov and N.A. Lebedev, Functions of a complex variable, constructive theory, Iliffe Books, London, 1968.
[3] L.N. Trefethen, Near-circularity of the error curve in complex Chebyshev approximation, J. Approx. Theory **31**, 344-367 (1981).

Solution of Trefethen's Problem 6

J. Boersma and F.W. Steutel
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Trefethen's Problem 6 as proposed in [1] reads:
A flea starts at (0,0) on the infinite 2D integer lattice and executes a biased random walk: At each step it hops north or south with probability 1/4 , east with probability $1/4+\varepsilon$, and west with probability $1/4-\varepsilon$. The probability that the flea returns to (0,0) sometime during its wanderings is 1/2. What is ε?

Solution. Let $u(m,n)$, $m,n \in \mathbb{Z}$, satisfy the partial difference equation

$$\begin{aligned} u(m,n) &= \delta_{m0}\delta_{n0} + (1/4+\varepsilon)u(m+1,n) + (1/4-\varepsilon)u(m-1,n) \\ &\quad + \tfrac{1}{4}u(m,n+1) + \tfrac{1}{4}u(m,n-1), \end{aligned}$$

where $\delta_{00} = 1$, $\delta_{m0} = 0$ for $m \neq 0$. By means of a generating-function technique it is found that

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} u(m,n) e^{imx} e^{iny} = \frac{2}{2-\cos x + 4i\varepsilon \sin x - \cos y} ,$$

whereupon

$$u(0,0) = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{dx\, dy}{2-\cos x + 4i\varepsilon \sin x - \cos y} .$$

By carrying out the integration over y one is led to

$$u(0,0) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{dx}{\sqrt{(1-\cos x + 4i\varepsilon \sin x)(3-\cos x + 4i\varepsilon \sin x)}} .$$

Substitute $e^{ix} = z$, then the latter integral transforms into

$$\begin{aligned} u(0,0) &= \frac{2}{\pi i} \int_{|z|=1} \frac{dz}{\sqrt{[(-1+4\varepsilon)z^2 + 2z - (1+4\varepsilon)][(-1+4\varepsilon)z^2 + 6z - (1+4\varepsilon)]}} \\ &= \frac{2}{\pi i (1-4\varepsilon)} \int_{|z|=1} \frac{dz}{\sqrt{(z-z_1)(z-1)(z_2-z)(z_3-z)}} , \end{aligned}$$

where $z_1 = (1+4\varepsilon)/(3+\sqrt{8+16\varepsilon^2})$ (with $0 < z_1 < 1$), $z_2 = (1+4\varepsilon)/(1-4\varepsilon)$, $z_3 = (3+\sqrt{8+16\varepsilon^2})/(1-4\varepsilon)$ (with $1 < z_2 < z_3$). Deform the integration contour $|z| = 1$ into a loop around the branch cut $[z_1, 1]$, then

$$u(0,0) = \frac{4}{\pi(1-4\varepsilon)} \int_{z_1}^{1} \frac{dx}{\sqrt{(x-z_1)(1-x)(z_2-x)(z_3-x)}} \, .$$

The latter integral can be reduced to a complete elliptic integral of the first kind, $K(k)$, by means of Byrd and Friedman [2, form. 253.00]:

$$u(0,0) = \frac{4\pi^{-1}}{(1+8\varepsilon^2+4\varepsilon\sqrt{2+4\varepsilon^2})^{1/2}} \, K(k),$$

where

$$k = \frac{\sqrt{1-16\varepsilon^2}}{1+8\varepsilon^2+4\varepsilon\sqrt{2+4\varepsilon^2}} \, .$$

Let $p(\varepsilon)$ denote the probability that the flea returns to (0,0) sometime. Then $p(\varepsilon) = [u(0,0)-1]/u(0,0)$, according to McCrea and Whipple [3]. The equation to be solved, i.e. $p(\varepsilon) = 1/2$, is equivalent to $u(0,0) = 2$. By use of Mathematica the solution is found as

$$\varepsilon = 0.0619139544739909$$

in 15 significant digits.

References

[1] L.N. Trefethen, The hundred-dollar, hundred-digit challenge problems, SIAM News, Vol. 35, Number 1, p. 3, January/February 2002.

[2] P.F. Byrd and M.D. Friedman, Handbook of elliptic integrals for engineers and physicists, Springer, Berlin, 1954.

[3] W.H. McCrea and F.J.W. Whipple, Random paths in two and three dimensions, Proc. Royal Soc. Edinburgh **A60**, 281-298 (1940).

Solution of Trefethen's Problem 7

J.K.M. Jansen

Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Trefethen's Problem 7 as proposed in [1] reads:
Let A be the $20,000 \times 20,000$ matrix whose entries are zero everywhere except for the primes $2, 3, 5, 7, \ldots, 224737$ along the main diagonal and the number 1 in all the positions a_{ij} with $|i - j| = 1, 2, 4, 8, \ldots, 16384$. What is the (1,1) entry of A^{-1}?

Solution. The (1,1) entry of A^{-1} is the first element of the solution vector $\mathbf{x}$ of the system of equations $A\mathbf{x} = \mathbf{b}$, where the vector $\mathbf{b}$ has first element 1, followed by 19,999 zeros. The system has been solved by 5 iterative methods taken from [2] and implemented in Matlab. As a result it was found that

$$\text{(1,1) entry of } A^{-1} = 0.725078346268,$$

to 12 decimal places.

References
[1] L.N. Trefethen, The hundred-dollar, hundred-digit challenge problems, SIAM News, Vol. 35, Number 1, p. 3, January/February 2002.
[2] R. Barrett *et al.*, Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods, SIAM, Philadelphia, 1994.

Solution of Trefethen's Problem 8

J. Boersma and J.K.M. Jansen
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Trefethen's Problem 8 as proposed in [1] reads:
A square plate $[-1,1]\times[-1,1]$ is at temperature $u=0$. At time $t=0$ the temperature is increased to $u=5$ along one of the four sides, and heat then flows into the plate according to $u_t=\Delta u$. When does the temperature reach $u=1$ at the center of the plate?

Solution. Introduce the function

$$T(x,y,t)=\frac{1}{4}\,[u(x,y,t)+u(-y,x,t)+u(-x,-y,t)+u(y,-x,t)].$$

Then $T(0,0,t)=u(0,0,t)$, and $T(x,y,t)$ is the solution of the symmetric heat conduction problem

$$T_t=\Delta T;\quad T=0 \text{ at } t=0;\quad T=5/4 \text{ along all four sides for } t>0.$$

Clearly, the stationary temperature is $\lim_{t\to\infty} T(x,y,t)=5/4$. Next, by use of the method of separation of variables the solution for $T(x,y,t)$ is found as

$$T(x,y,t)=\frac{5}{4}-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}\cos((m+1/2)\pi x)\cos((n+1/2)\pi y)\cdot \\ \cdot\exp[-\{(m+1/2)^2+(n+1/2)^2\}\pi^2 t],$$

where the coefficients A_{mn} are determined by the initial condition $T=0$ at $t=0$, leading to

$$A_{mn}=\frac{5}{\pi^2}\,\frac{(-1)^{m+n}}{(m+1/2)(n+1/2)}\;.$$

The temperature at the center of the plate is now given by

$$u(0,0,t)=T(0,0,t)=\frac{5}{4}-\frac{5}{\pi^2}\left\{\sum_{m=0}^{\infty}\frac{(-1)^m}{m+1/2}\,\exp[-(m+1/2)^2\pi^2 t]\right\}^2\;.$$

The time t_0 at which $u(0,0,t_0)=1$ is found from the equation

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{m+1/2} \exp[-(m+1/2)^2\pi^2 t_0] = \frac{\pi}{2\sqrt{5}} \, .$$

The series converges extremely rapidly and a restriction of the summation to two terms already yields the result

$$t_0 = 0.42401138703$$

with 11 correct digits.

Reference

[1] L.N. Trefethen, The hundred-dollar, hundred-digit challenge problems, SIAM News, Vol. 35, Number 1, p. 3, January/February 2002.

Solution of Trefethen's Problem 9

J. Boersma and J.K.M. Jansen
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Trefethen's Problem 9 as proposed in [1] reads:
The integral $I(\alpha) = \int_0^2 [2+\sin(10\alpha)]\, x^\alpha \sin(\alpha/(2-x))\, dx$ depends on the parameter α. What is the value $\alpha \in [0,5]$ at which $I(\alpha)$ achieves its maximum?

Solution. Write $I(\alpha) = [2+\sin(10\alpha)]\, J(\alpha)$, then the integral $J(\alpha)$ is calculated in the following manner. Substitute $x = 2-1/t$, then $J(\alpha)$ transforms into

$$J(\alpha) = \int_{1/2}^{\infty} \left(\frac{2t-1}{t}\right)^{\alpha} \frac{\sin(\alpha t)}{t^2}\, dt.$$

By contour integration in the complex z-plane one has

$$\int_{1/2}^{\infty} \left(\frac{2t-1}{t}\right)^{\alpha} \frac{e^{i\alpha t}}{t^2}\, dt = \int_C \left(\frac{2z-1}{z}\right)^{\alpha} \frac{e^{i\alpha z}}{z^2}\, dz + \int_{i/2}^{i\infty} \left(\frac{2z-1}{z}\right)^{\alpha} \frac{e^{i\alpha z}}{z^2}\, dz,$$

where C is the arc of the circle $|z| = 1/2$ from $z = 1/2$ to $z = i/2$. In the two integrals on the right, parametrize $z = \frac{1}{2}e^{it}$, $0 \le t \le \pi/2$, and $z = iy$, $y \ge 1/2$. By taking imaginary parts it follows that

$$\begin{aligned} J(\alpha) &= 2^{2\alpha+1} \int_0^{\pi/2} (\sin(t/2))^\alpha \exp[-\tfrac{1}{2}\alpha \sin t] \cos(\tfrac{1}{2}\alpha(\pi + \cos t - t) - t)\, dt \\ &\quad - 2^\alpha \int_{1/2}^{\infty} (y^2+1/4)^{\alpha/2} \cos(\alpha \arctan(1/(2y)))\, \frac{e^{-\alpha y}}{y^{\alpha+2}}\, dy. \end{aligned}$$

Both integrals on the right can be numerically evaluated by means of Mathematica. A plot of $I(\alpha)$ for $0 \le \alpha \le 5$ shows that $I(\alpha)$ achieves its maximum near $\alpha = 0.8$. Next, determine $J'(\alpha)$ by analytical differentiation of $J(\alpha)$ with respect to α. Solve the equation $I'(\alpha) = 0$ numerically by use of Mathematica. Thus it is found that $I(\alpha)$ achieves its maximum at

$$\alpha = 0.78593367435,$$

correct to 11 decimal places.

Reference

[1] L.N. Trefethen, The hundred-dollar, hundred-digit challenge problems, SIAM News, Vol. 35, Number 1, p. 3, January/February 2002.

Solution of Trefethen's Problem 10

J. Boersma
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Trefethen's Problem 10 as proposed in [1] reads:
A particle at the center of a 10×1 rectangle undergoes Brownian motion (i.e., 2D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?

Solution. The analytical part of the solution is found by an approach taken from Zauderer [2, Section 1.3]. Generalize the problem as follows: A particle at $(x,y) = (0, K'/2)$ undergoes Brownian motion in the rectangular domain $D = \{(x,y) \,|\, -K < x < K,\, 0 < y < K'\}$ till it hits the boundary of D; here, $K = K(k)$ and $K' = K(k')$ are complete elliptic integrals of the first kind with modulus k and $k' = \sqrt{1-k^2}$. The question is to determine the probability p, say, that the particle hits at one of the ends $x = \pm K$ rather than at one of the sides $y = 0$ or $y = K'$.
Introduce the Green's function $G(x,y)$ with singularity at $(x,y) = (0, K'/2)$, specified by

$$\begin{aligned} \Delta G &= G_{xx} + G_{yy} = -\delta(x)\,\delta(y - K'/2) \text{ in } D, \\ G(x,y) &= 0 \text{ along the boundary of } D. \end{aligned}$$

Then, according to Zauderer [2, Section 1.3], the probability p is given by

$$p = -2 \int_0^{K'} \frac{\partial G}{\partial x}(K,y)\,dy.$$

The Green's function $G(x,y)$ is determined by use of the conformal mapping $\zeta = \operatorname{sn}(z,k) = \operatorname{sn} z$, where $z = x + iy$; for the theory of the Jacobian elliptic function $\operatorname{sn} z$, see Byrd and Friedman [3, pp. 18-28]. The mapping $\zeta = \operatorname{sn} z$ transforms D into the half-plane $\operatorname{Im} \zeta > 0$, and sends $z = iK'/2$ to $\zeta = \operatorname{sn}(iK'/2) = i/\sqrt{k}$ by [3, form. 129.50, 122.11]. Then it easily follows that $G(x,y)$ is given by

$$G(x,y) = -\frac{1}{2\pi} \log \left| \frac{\operatorname{sn} z - i/\sqrt{k}}{\operatorname{sn} z + i/\sqrt{k}} \right|, \quad z = x + iy.$$

Determine the harmonic conjugate

$$H(x,y) = -\frac{1}{2\pi}\arg\left(\frac{\operatorname{sn} z - i/\sqrt{k}}{\operatorname{sn} z + i/\sqrt{k}}\right), \quad z = x + iy,$$

which is related to $G(x,y)$ by the Cauchy-Riemann equations $\partial G/\partial x = \partial H/\partial y$, $\partial G/\partial y = -\partial H/\partial x$. Then the probability p can be evaluated as

$$p = -2\int_0^{K'} \frac{\partial H}{\partial y}(K,y)\,dy = -2H(K,K') + 2H(K,0).$$

By use of the known values $\operatorname{sn} K = 1$, $\operatorname{sn}(K+iK') = 1/k$, from [3, form. 122.02, 122.07], we find the following expressions for p:

$$p = \frac{2}{\pi}[\arctan(1/\sqrt{k}) - \arctan(\sqrt{k})] = 1 - \frac{4}{\pi}\arctan(\sqrt{k}) = \frac{4}{\pi}\arctan\left(\frac{1-\sqrt{k}}{1+\sqrt{k}}\right).$$

By means of Whittaker and Watson [4, p. 486] the rightmost expression can be expanded as

$$p = \frac{4}{\pi}\arctan\left(\frac{2\sum_{n=0}^{\infty}(q')^{(2n+1)^2}}{1+2\sum_{n=1}^{\infty}(q')^{4n^2}}\right) = \frac{8}{\pi}\,q' + O({q'}^3),$$

where $q' = e^{-\pi K/K'}$.
To solve the original problem, the modulus k is to be chosen such that $2K/K' = 10$ or $K(k)/K(k') = 5$. The latter equation has the solution $k = \sqrt{1-k_{25}^2}$, where k_{25} is a so-called singular modulus; generally, the singular modulus k_N satisfies $K(\sqrt{1-k_N^2})/K(k_N) = \sqrt{N}$. It so happens that the value of k_{25} is explicitly known from Borwein and Borwein [5, p. 162]: $k_{25} = (\sqrt{5}-2)(3-2\cdot 5^{1/4})/\sqrt{2}$. Then the probability p for a 10×1 rectangle is found to be

$$p = 1 - \frac{4}{\pi}\arctan(\sqrt[4]{1-k_{25}^2}) = 3.8375879792512261034 \times 10^{-7}$$

to 20 significant digits. In the present case $q' = e^{-5\pi}$ and the corresponding approximation

$$p \approx \frac{8}{\pi}\,e^{-5\pi} = 3.837587979251342\ldots \times 10^{-7}$$

has already 13 correct digits.
Several singular moduli k_N are known from [5]. This permits an easy calculation of the probability p for a $2\sqrt{N} \times 1$ rectangle.

References

[1] L.N. Trefethen, The hundred-dollar, hundred-digit challenge problems, SIAM News, Vol. 35, Number 1, p. 3, January/February 2002.
[2] E. Zauderer, Partial differential equations of applied mathematics, Second edition, Wiley, New York, 1989.
[3] P.F. Byrd and M.D. Friedman, Handbook of elliptic integrals for engineers and physicists, Springer, Berlin, 1954.
[4] E.T. Whittaker and G.N. Watson, A course of modern analysis, Fourth edition, Cambridge University Press, Cambridge, 1958.
[5] J.M. Borwein and P.B. Borwein, Pi and the AGM, Wiley, New York, 1987.