

**Problem:**

Find the sum of this slowly converging series to 10 or more digits:

$$S = \sum_{n=1}^{\infty} \frac{1}{f(n)} \tag{1}$$

where  $f(1) = 1$ ,  $f(2) = 2$ , and for each  $n > 2$  define  $f(n) = n f(d)$ , where  $d$  is the number of base 2 digits of  $n$ .

**Solution:**

The values of the first few  $f$ 's are 1, 2, 6, 24, 30, 36, 42, 192, 216, 240, 264, 288, 312, 336, 360, 480. Directly adding the series does not tell much – the sums of the first  $10^2$ ,  $10^3$ , and  $10^4$  terms are about 1.88, 1.89, 1.90. Extrapolation does not really help, because for long stretches the partial sums do not appear to be converging (see below), and at each power of 2 the value of  $d$  changes, causing a non-smooth jump in the partial sums.

The fact that  $d$  stays the same between powers of 2 leads to this interpretation of the sum.

$$S = \frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{4f(3)} + \frac{1}{5f(3)} + \frac{1}{6f(3)} + \frac{1}{7f(3)} + \frac{1}{8f(4)} + \frac{1}{9f(4)} \dots$$

since 4 through 7 each have three digits base 2, 8 through 15 each have four, and so on.

Let  $H(k)$  denote the  $k$ th harmonic number,  $H(k) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{k}$ . Then  $S$  can be written

$$\begin{aligned} S &= 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{f(3)} \left( H(7) - H(3) \right) + \frac{1}{f(4)} \left( H(15) - H(7) \right) + \dots \\ &= \frac{5}{3} + \sum_{n=3}^{\infty} \frac{H(2^n - 1) - H(2^{n-1} - 1)}{f(n)}. \end{aligned} \tag{2}$$

When  $n$  gets large enough we can evaluate these harmonic numbers using an asymptotic formula such as  $H(k) \sim \ln(k) + \gamma + \frac{1}{2k} - \frac{1}{12k^2} + \frac{1}{120k^4}$ . Here  $\gamma \approx 0.57721566490153286060651209$  is Euler's constant, and the error in this formula is less than  $1/(252k^6)$ . So if  $k > 1000$ , then this formula gives  $H(k)$  correct to at least 20 significant digits.

Adding 100 terms of sum (2) is equivalent to adding  $2^{100} - 1 = 1.3 \times 10^{30}$  terms of sum (1). Check a few of these partial sums for convergence ( $n$  is the number of terms in sum (1)):

$n = 2^{100} - 1$	sum = 1.94161
$n = 2^{101} - 1$	sum = 1.94177
$n = 2^{102} - 1$	sum = 1.94193
$n = 2^{200} - 1$	sum = 1.94717
$n = 2^{300} - 1$	sum = 1.94846

$$\begin{aligned} n = 2^{1000} - 1 & \quad \text{sum} = 1.95221 \\ n = 2^{2000} - 1 & \quad \text{sum} = 1.95403 \end{aligned}$$

This *is* a slowly convergent series. After adding  $10^{602}$  terms, we seem to have about one significant digit of accuracy.

Looking more closely at the terms of (2) shows that  $H(2^n - 1) - H(2^{n-1} - 1)$  converges to  $\ln(2)$  as  $n$  increases, so eventually these terms are approaching zero no faster than the terms in (1).

Since the numerators of (2) approach a constant, we can combine the two sums in a way that creates a third sum with a reasonable rate of convergence.

$$\begin{aligned} S - \ln(2)S &= \frac{5}{3} + \sum_{n=3}^{\infty} \frac{H(2^n - 1) - H(2^{n-1} - 1)}{f(n)} - \ln(2) \left( \frac{3}{2} + \sum_{n=3}^{\infty} \frac{1}{f(n)} \right) \\ &= \frac{5}{3} - \frac{3\ln(2)}{2} + \sum_{n=3}^{\infty} \frac{H(2^n - 1) - H(2^{n-1} - 1) - \ln(2)}{f(n)} \end{aligned}$$

This leads to

$$S = \left( \frac{1}{1 - \ln(2)} \right) \left( \frac{5}{3} - \frac{3\ln(2)}{2} + \sum_{n=3}^{\infty} \frac{H(2^n - 1) - H(2^{n-1} - 1) - \ln(2)}{f(n)} \right) \quad (3)$$

From the asymptotic formula for  $H(k)$  we get  $H(2^n - 1) - H(2^{n-1} - 1) - \ln(2) = 2^{-n-1} + O(2^{-2n-2})$ , so 50 terms should be enough to give 15 significant digit accuracy. Check a few of these partial sums for convergence ( $n$  is the number of terms in sum (3)):

$$\begin{aligned} n = 5 & \quad \text{sum} = 2.08530047223333 \\ n = 10 & \quad \text{sum} = 2.08637209927587 \\ n = 20 & \quad \text{sum} = 2.08637766264294 \\ n = 30 & \quad \text{sum} = 2.08637766500453 \\ n = 40 & \quad \text{sum} = 2.08637766500599 \\ n = 50 & \quad \text{sum} = 2.08637766500599 \end{aligned}$$