SIAM 100-DIGIT CHALLENGE EXTRA PROBLEM NUMBER 7

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1. Problem

We are asked to obtain a very high accuracy estimate of the integral

$$I = \int_{-1}^{1} dx \exp\left\{x + \sin\left(e^{e^{e^{x+1/3}}}\right)\right\}. \tag{1}$$

As the problem is written, this would be very difficult, since the integrand is extremely oscillatory over most of the range; any numerical quadrature method would require an inordinately large number of points to obtain an accurate integration. One method that is often used for integrals of this type is that of rewriting the integral as a sum of sub-integrals taken over single periods of the integrand, either between zeros or extrema (see [1]). The integral over each segment can be calculated accurately by simple quadrature methods, and summing the series yields the desired integral.

However, I have found that a different approach works well for the integral in question. I convert the integral into a sum of simpler integrals, and then perform these by using a contour in the complex plane that allows one to minimize the effects of the rapid oscillations. First, I present a very simple approximation of the integral that will let us know, roughly, what value to expect.

1.1. **Approximate Treatment.** Figure 1 shows the integrand, plotted in blue. The oscillations are so rapid for $x \geq 0$ that they cannot be seen in the plot. Clearly, these rapid oscillations should contribute a relatively small amount to the integral, after we subtract the local mean value of the integrand. The red curve in Figure 1 shows a plot of Ce^x, where C is the mean value of $e^{\sin(x)}$ over one period, which is

$$C = \frac{1}{2\pi} \int_{0}^{2\pi} dx\ e^{\sin(x)} = I_0(1) \approx 1.26,$$

where $I_0(x)$ is the modified Bessel function of the first kind.

We can then roughly approximate I by finding the integral of the function shown by the red curve,

$$I \approx C \int_{-1}^{1} dx\ e^x \approx 2.98. \tag{2}$$

This approximate value will turn out to be surprisingly close to the correct value obtained in the next section.

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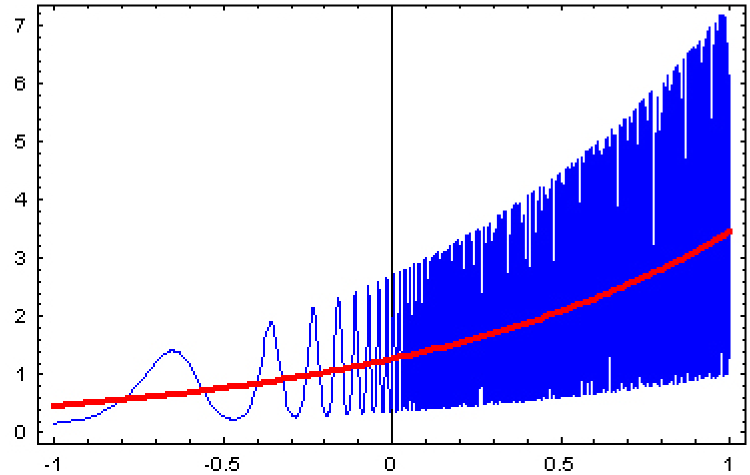

FIGURE 1. Blue - original integrand; red - approximate integrand

2. Solution

The change of variable

$$\alpha = \exp\exp\exp(x + 1/3)$$

yields

$$I = c\int_a^b d\alpha \frac{e^{\sin\alpha}}{\alpha\log\alpha},$$

where $c = e^{-1/3}$, $a = \exp(\exp(\exp(-2/3)))$, and $b = \exp(\exp(\exp(4/3)))$.

The integrand oscillates much more slowly now, but of course is taken over a much larger interval. We take advantage of the boundedness of the sin() function, implying that a relatively small number of terms will be needed to obtain an accurate value for $e^{\sin(\alpha)}$, and expand the exponential in the integrand in a Taylor series, giving

$$I = c\int_a^b d\alpha \frac{1}{\alpha\log\alpha}\sum_{n=0}^{\infty}\frac{\sin^n\alpha}{n!}.$$

In what follows, we will denote by I_N the above sum when its upper limit is set to N; we will see how the number of correct digits varies as N is increased.

In order to simplify the integrand further, we write the sin() function in terms of complex exponentials, $\sin(\alpha) = \frac{e^{i\alpha}-e^{-i\alpha}}{2i}$. We then use the binomial theorem to expand the powers of the sum of the two exponentials to obtain

$$I_N = c\int_a^b d\alpha \frac{1}{\alpha\log\alpha}\sum_{n=0}^{N}\sum_{k=0}^{n}\frac{e^{i\alpha(n-2k)}}{(2i)^n}\frac{(-1)^k}{n!}\binom{n}{k}. \tag{3}$$

We then interchange the order of summation and integration, which presents no problem since both are finite, leading to

$$I_N = c\sum_{n=0}^{N}\sum_{k=0}^{n}\frac{1}{(2i)^n}\frac{(-1)^k}{n!}\binom{n}{k}R_{n-2k}, \tag{4}$$

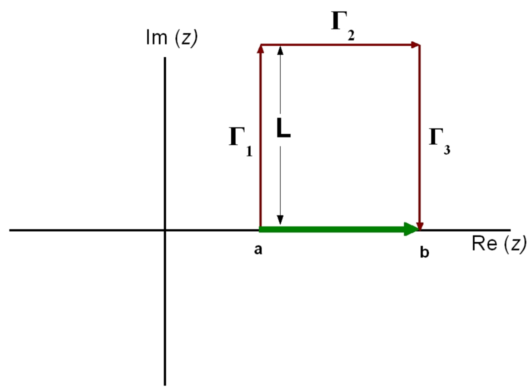

FIGURE 2. Integration contour in the complex plane (red arrows) for numerical computation of integrals R_m in Equation 5

where

$$R_m = \int_a^b d\alpha \frac{e^{im\alpha}}{\alpha \log \alpha}. \tag{5}$$

This last integral can be performed analytically for $m = 0$, giving $R_0 = \log\log(b) - \log\log(a)$. For other values of m, the integrand is still oscillatory along the real axis, again making it difficult to estimate numerically. In order to avoid the oscillatory behavior we deform the path into a contour in the complex plane, shown in Figure 2 by the red arrows, denoted by $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$. (This is allowable by Cauchy's theorem since the integrand is analytic in the region in which we are interested.) We will show below that by picking a large enough value for L, the height of the integration path Γ_2 above the real axis, we may easily compute the integrals R_m to high accuracy.

The integrand now decays exponentially on the two vertical portions of the contour (Γ_1 and Γ_3), and numerical integration using the double-exponential transform ([1]) can quickly yield results valid to 50 or more digits. We have used *Mathematica's* implementation of this method to do this computation. As stated above, the oscillatory portion of the contour parallel to the real axis can be made negligible by picking L large enough. After choosing the contour, we precompute values of R_m, for $m \geq 0$, and use $R_{-m} = R_m^*$. Performing the sum in Equation 4 using high-precision arithmetic (again in *Mathematica*) yields the estimate of the integral.

We can bound the contribution of the portion of the contour parallel to the real axis, Γ_2, as follows. We have

$$\begin{aligned} \left| \int_{\Gamma_2} \frac{e^{ikz}}{z \log z} \right| &<= \int_{\Gamma_2} dz \left| \frac{e^{ikz}}{z \log z} \right| = e^{-kL} \int_{\Gamma_2} dz \left| \frac{1}{z \log z} \right| \\ &= e^{-kL} \int_a^b dt \frac{1}{\sqrt{(t^2 + L^2) \log(t + iL) \log(t - iL)}} \\ &< e^{-kL} \int_a^b dt \frac{1}{t \log t} = e^{-kL} \log \left[\frac{\log(b)}{\log(a)} \right] \approx 3.3 e^{-kL} \end{aligned}$$

(This is a very rough bound, ignoring the cancellation from oscillations.)

Therefore, for example, choosing L to be 500 will reduce the contribution from Γ_2 to less than 10^{-217}. Note that the contribution is actually much smaller, since the oscillations tend to cancel. In addition, L could be made smaller as k increases, although I have chosen to leave it constant in my computations.

After developing this solution, I became aware of the paper [2] that presents a general numerical method for evaluating oscillatory integrals by rewriting them as contour integrals. The authors use the terminology *numerical steepest descent* for their method. In fact, their first example is for a Fourier integral that generalizes R_m, and indeed they utilize the contour shown in Figure 2 (with L taken to infinity).

Finally, the following *Mathematica* code shows my implementation of the above algorithm, and the table shows the value of the estimate I_N as a function of N, the maximum number of terms used in the sum (see Equation 4), for $L = 500$. Only the correct digits are shown, so we may easily see how the number of correct digits in the answer increases with N. Note that $N = 72$ yields 106 correct digits, and that the rough estimate obtained in Section 1.1 comes quite close to the right answer.

```
a = N[Exp[Exp[Exp[-2/3]]],100]
b = N[Exp[Exp[Exp[4/3]]],100]
fval500 = Table[NIntegrate[ N[1/(z*Log[z]) Exp[I k z],100],
          {z,a,a+500*I},PrecisionGoal->80,
          WorkingPrecision->100,Method->DoubleExponential]
        + NIntegrate[N[1/(z*Log[z]) Exp[I k z],100],{z,b+500*I,b},
          PrecisionGoal->80,WorkingPrecision->100,
          Method->DoubleExponential] ,
          {k,1,81,1}] ;

fv[k_] := If[k>0,fval500[[k]],Conjugate[fval500[[-k]]]];
fv[0] = Log[Log[b]] - Log[Log[a]];
WorkingPrecision->100;
c = N[Exp[-1/3],100];

For[NN=3,NN<=80,NN++,
For[sum=0;n=0,n<=NN,n++,
   For[k=0,k<=n,k++,
      sum = sum + 1/n! (-1)^k 1/(2 I)^n
      Binomial[n,k] fv[n-2*k]]];
      val[NN] = c*sum]

Table[{j,Chop[val[j]]},{j,3,80}] //TableForm

 3 2.9
 4 2.99
 5 2.99
 6 2.9963
 7 2.9963
 8 2.996339
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 9 2.996339
10 2.99633981
11 2.99633981
12 2.9963398119
13 2.99633981197
14 2.9963398119765
15 2.9963398119765
16 2.996339811976535
17 2.9963398119765353
18 2.99633981197653537
19 2.996339811976535379
20 2.996339811976535379182
21 2.996339811976535379182
22 2.99633981197653537918286
23 2.99633981197653537918286
24 2.99633981197653537918286155
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26 2.99633981197653537918286155623
27 2.99633981197653537918286155623
28 2.996339811976535379182861556235
29 2.99633981197653537918286155623550
30 2.99633981197653537918286155623550188
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91442762852494355617494421407018160 8302
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3. Comments by Folkmar Bornemann

After reading my solution, Dr. Bornemann made the excellent point that the contour height L can just as well be taken to infinity, since *Mathematica's* double-exponential quadrature method is quite capable of handling an infinite integration range. He also provided the following very elegant *Mathematica* code implementing his suggestion:

```
a = Exp[Exp[Exp[-2/3]]]; b = Exp[Exp[Exp[4/3]]];
c = Exp[-1/3];

R[s_, m_, prec_] :=
   R[s, m, prec] = NIntegrate[N[Exp[I m s], Max[17, prec]]
         With[{z = s + I t},I c Exp[-m t]/(z Log[z]),
        {t, 0, Infinity},Method -> DoubleExponential,
        WorkingPrecision -> prec]

 R[m_?Positive,prec_] := R[a,m,prec]-R[b,m,prec];

 R[m_?Negative,prec_] := Conjugate[R[-m,prec]];

 R[0, prec_] = Integrate[c/(z Log[z]), {z, a, b}];
 II[N_, prec_] := Sum[1/(2 I)^n (-1)^k/n!
       Binomial[n, k] R[n - 2 k, prec], {n, 0, N}, {k, 0, n}]
       // Chop

 {#,II[#,$MachinePrecision]}&/@Range[13,17]
         //Transpose//Timing//TableForm

 0.18 Second
 13     2.99634
 14     2.99634
 15     2.99634
 16     2.99634
 17     2.99634


{#,II[#,100]}&/@Range[67,71]//Transpose//Timing//TableForm

46.447 Second
 67   2.996339811976535379182861556235501887668096456254829159481 60
88321749914427628524943556174944214069 30
 68   2.996339811976535379182861556235501887668096456254829159481 60
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8832174991442762852494355617494421407018
69 2.9963398119765353791828615562355018876680964562548291594816O
8832174991442762852494355617494421407018
70 2.9963398119765353791828615562355018876680964562548291594816O
8832174991442762852494355617494421407018
71 2.9963398119765353791828615562355018876680964562548291594816O
8832174991442762852494355617494421407018

References

[1] F. Bornemann, D. Laurie, S. Wagon, and J. Waldvogel, *The SIAM 100-Digit Challenge*, The Society for Industrial and Applied Mathematics, Philadelphia, 2004.

[2] D. Huybrechs and S. Vandewalle, *On the evaluation of highly oscillatory integrals by analytic continuation*, SIAM J. Numer. Anal., 44, (2006), pp. 1026-1048.

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